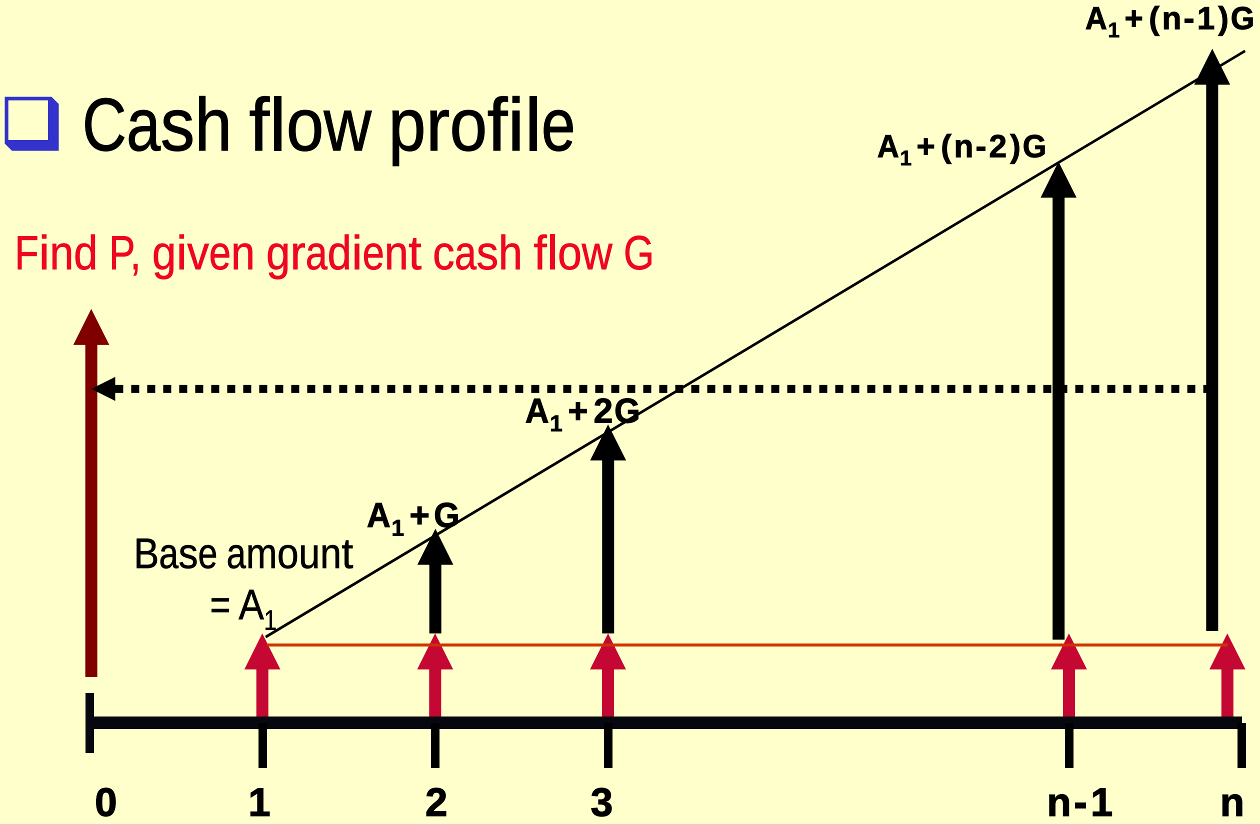
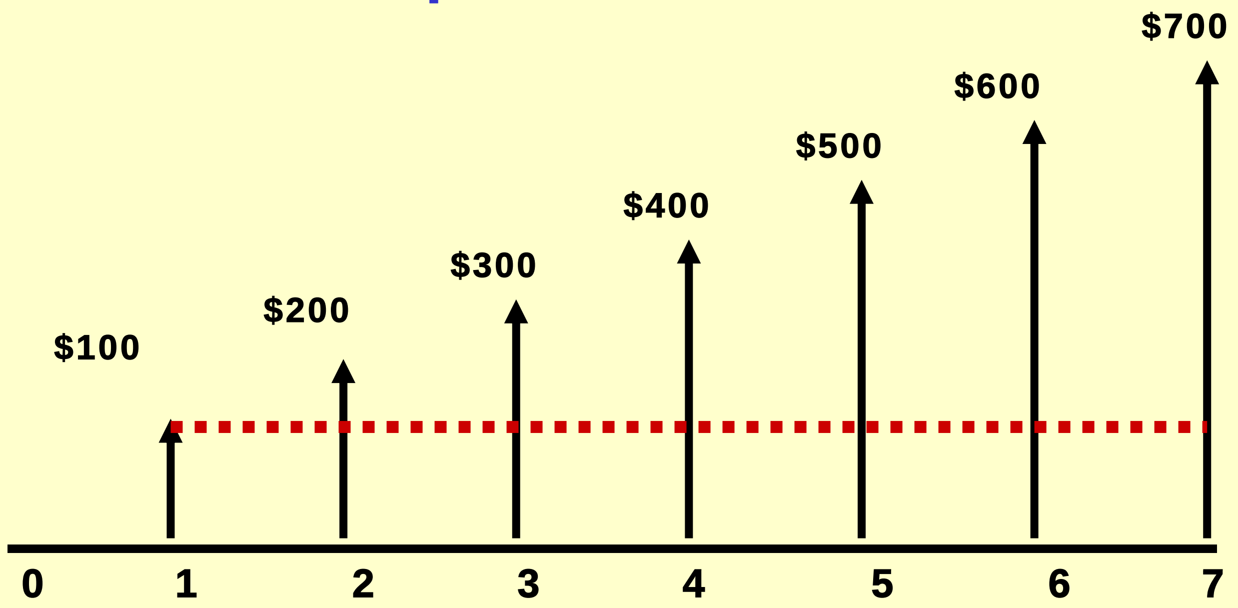
2.5 Arithmetic Gradient Factors (P/G) and (A/G)

- Cash flow profile



$$CF_n = A_1 \pm (n-1)G$$

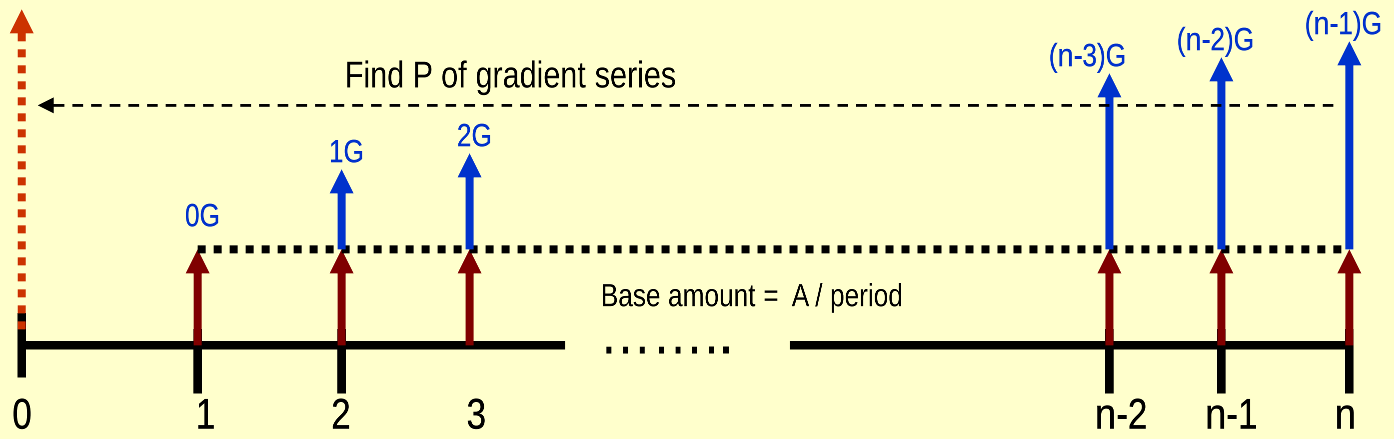
Gradient Example



Gradients have two components:

- The base amount (A_1 =$100) and the gradient (G=$100)

Gradient Components



- Present worth point is 1 period to the left of the 0G cash flow
- For present worth of the base amount, use the P/A factor (already known)
- For present worth of the gradient series, use the P/G factor (to be derived)

(P/G) Factor Derivation

$$P=G(P/F,i,2)+2G(P/F,i,3)+ \ldots + (n-2)G(P/F,i,n-1)+(n-1)G(P/F,i,n)\}$$

$$P=G\{(P/F,i,2)+2(P/F,i,3)+ \ldots + (n-2)(P/F,i,n-1)+(n-1)(P/F,i,n)\}$$

(P/G) Factor Derivation

(P/G) Factor Derivation

[2] – [1]

(P/G) Factor Derivation

$$Pi = G\left[\frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n} - \frac{n}{(1+i)^n}\right]$$

$$P = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]$$

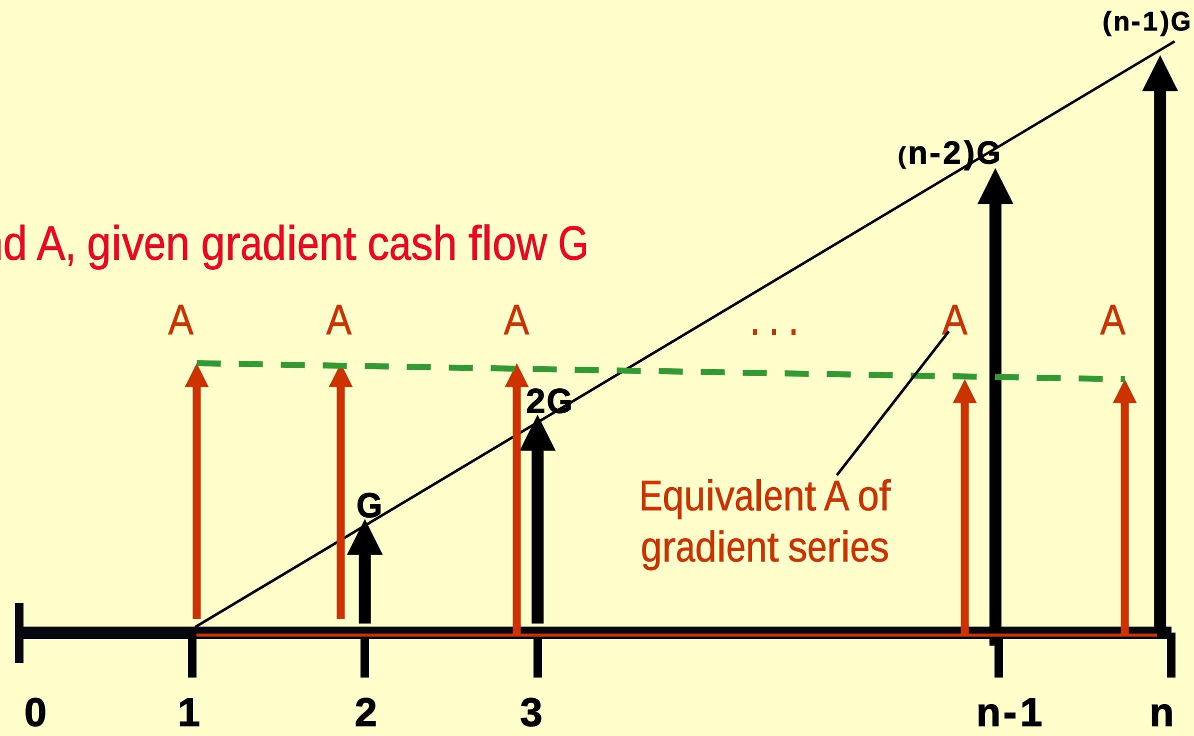
(P/G, i%, n) factor

Gradient Decomposition

- As we know, arithmetic gradients are comprised of two components
  1. Gradient component
  2. Base amount
- When working with a cash flow containing a gradient, the (P/G) factor is only for the gradient component
- Apply the (P/A) factor to work on the base amount component
- P = PW(gradient) + PW(base amount)

Use of the (A/G) Factor

Find A, given gradient cash flow G



$$A = G(A/G,i,n)$$

(A/G) Factor Derivation

- A/G converts a linear gradient to an equivalent annuity cash flow.

$$A = G(P/G,i,n)(A/P,i,n)$$

$$A = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

(A/G) Factor Derivation

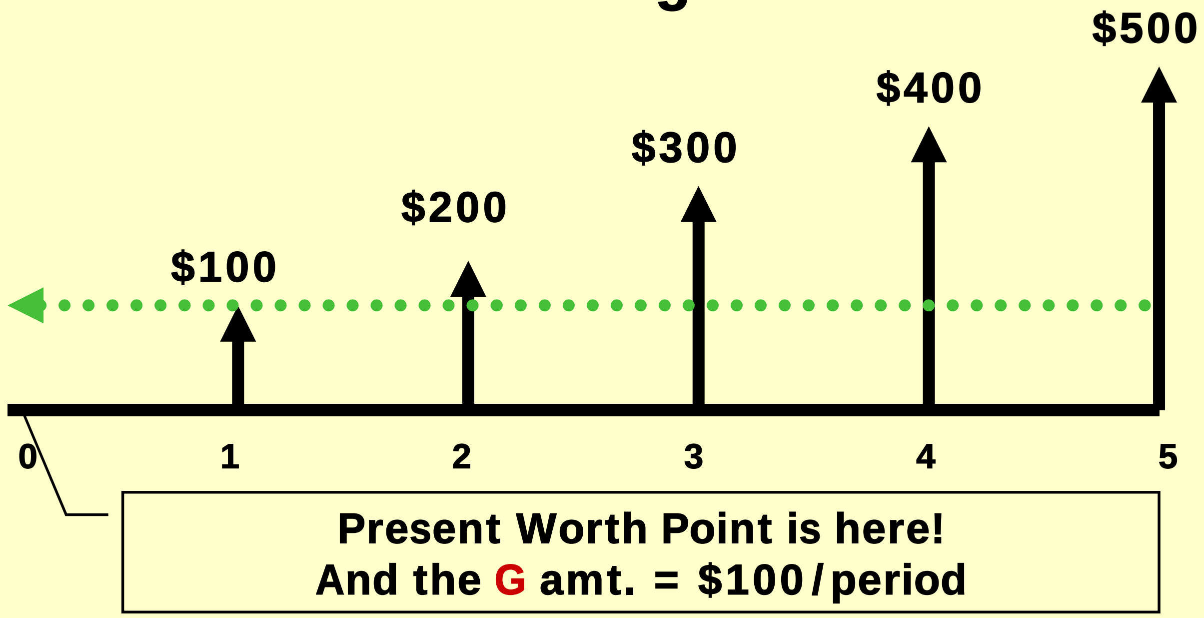
$$A = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

(A/G,i,n) =

$$A = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$$

Example

• **Consider the following cash flow**

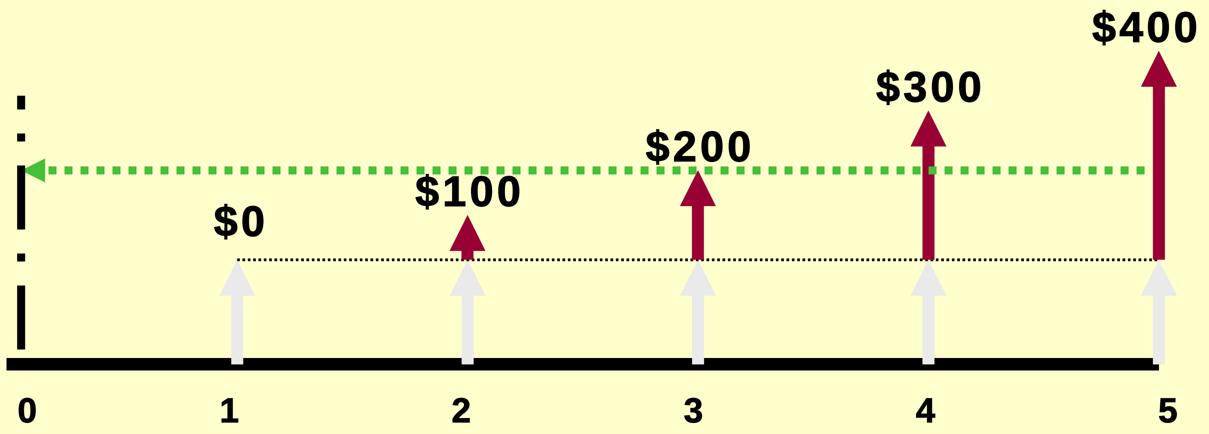


Find the present worth if i = 10%/yr; n = 5 yrs

Example

- **First, The Base Annuity of $100/period**

Example

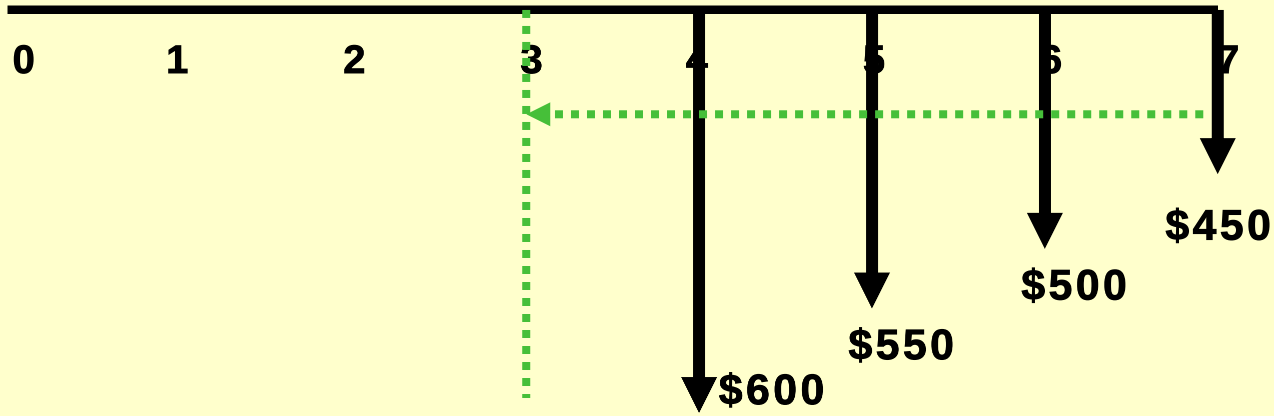


Example

- **$PW(10\%)_{\text{Base Annuity}}$ = $379.08**
- **$PW(10\%)_{\text{Gradient Component}}$ = $686.18**
- **Total PW(10%) = $379.08 + $686.18**
- **Equals <u>$1065.26</u>**
- **Note: The two sums occur at t = 0 and can be added together – concept of equivalence**

Shifted Gradient Example

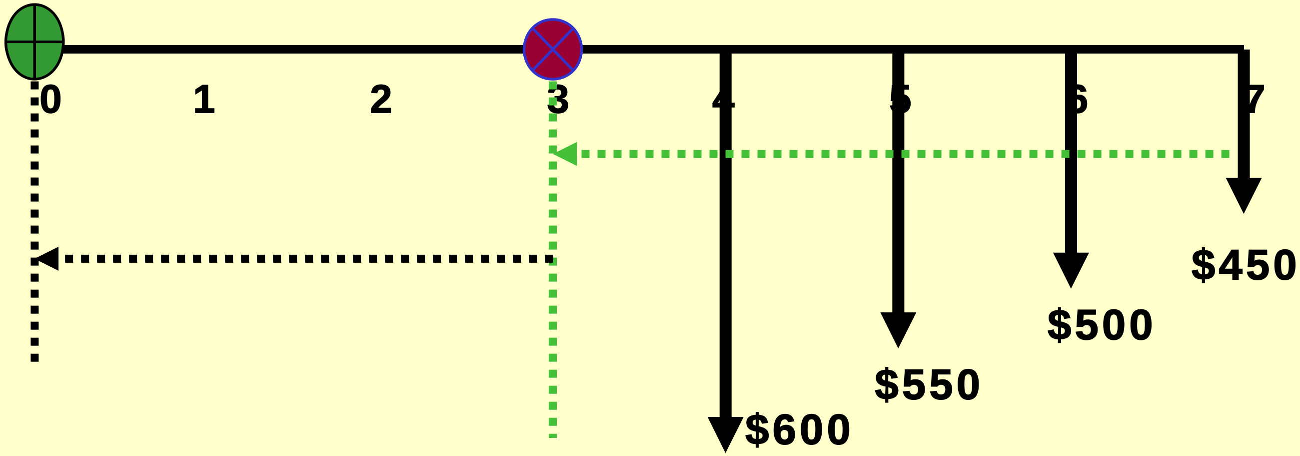
- **Consider the following Cash Flow**



1. **This is a "shifted" negative, decreasing gradient.**
2. **The PW point in time is at t = 3 (not t = o)**

Shifted Gradient Example

- Consider the following Cash Flow



- The PW @ t = 0 requires getting the PW @ t = 3;
- Then using the P/F factor move PW_3 back to t=0

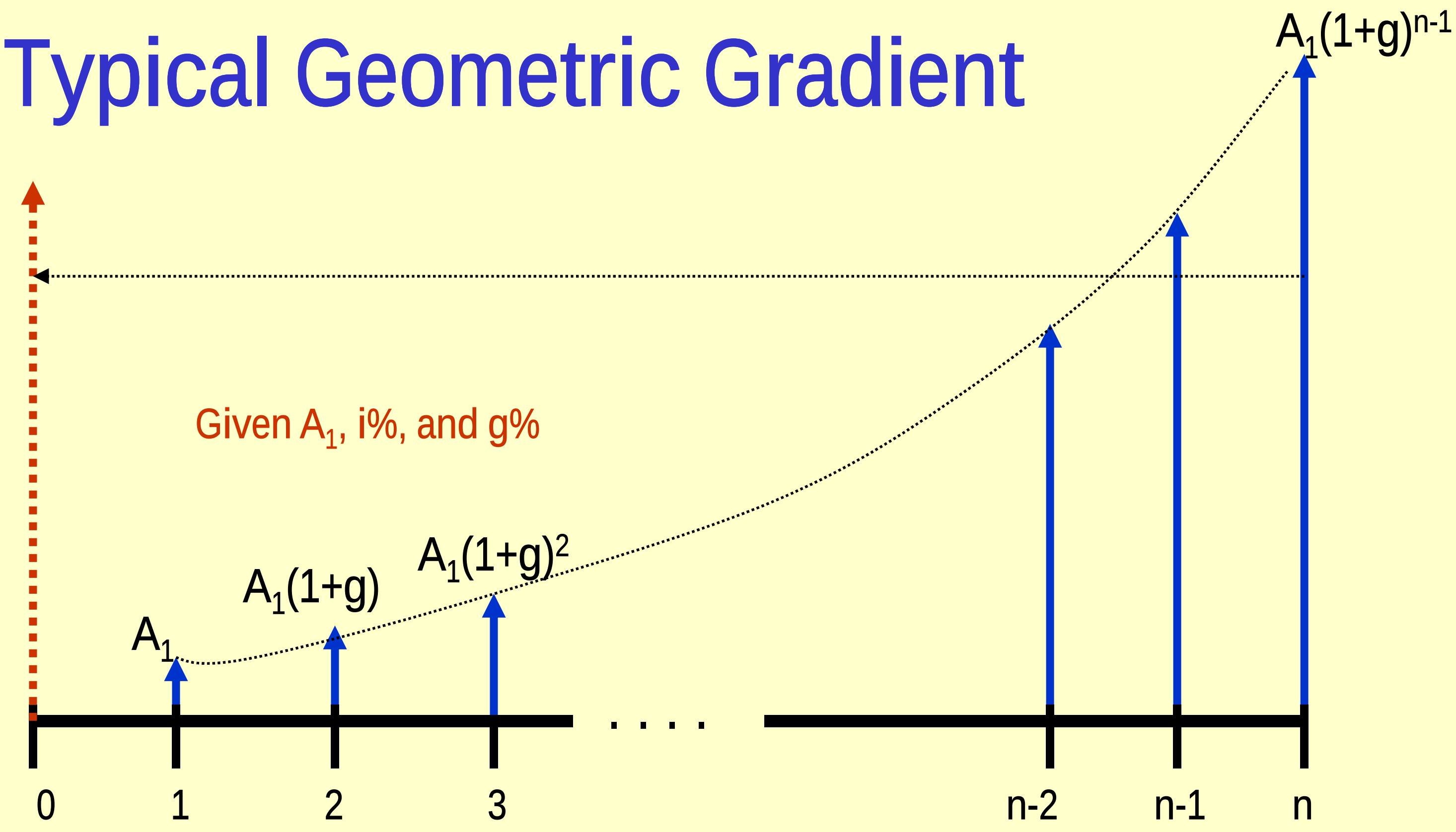
Shifted Gradient Example

2.6 Geometric Gradient Series Factor

- Geometric Gradient
  - Cash flow series that starts with a base amount A_1
  - Increases or decreases from period to period by a <u>constant percentage</u> amount
  - This uniform rate of change defines
    - <u>A GEOMETRIC GRADIENT</u>
    - Notation:

      g = the constant rate of change, in decimal form, by which future amounts increase or decrease from one time period to the next

Typical Geometric Gradient



$$P_g = \frac{A_1}{(1+i)^1} + \frac{A_1(1+g)}{(1+i)^2} + \frac{A_1(1+g)^2}{(1+i)^3} + \ldots + \frac{A_1(1+g)^{n-1}}{(1+i)^n}$$

Geometric Gradients Derivation

Geometric Gradients Derivation

$$P_g\left(\frac{1+g}{1+i}-1\right)=A_1\left[\frac{(1+g)^n}{(1+i)^{n+1}}-\frac{1}{1+i}\right]$$

Geometric Gradient: i ≠ *g* Case

$$P_g = A_1 \left[\frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g} \right] \quad g \neq i$$

- **This is the (P/A,*g*,i,n) factor and is valid if g is not equal to i.**

Geometric Gradient: i = *g* Case

$$P_g = A_1\left(\frac{1}{(1+i)}+\frac{1}{(1+i)}+\frac{1}{(1+i)}+\ldots+\frac{1}{(1+i)}\right)$$

$$P_g = \frac{nA_1}{(1+i)}$$

For the case i = g

Geometric Gradients: Summary

$$P_g = A_1 \left[\frac{1 - \left(\frac{1+g}{1+i} \right)^n}{i - g} \right] \quad g \neq i$$

Case: $g \neq i$

$$P_g = \frac{nA_1}{(1+i)}$$

Case: $g = i$

To use the (P/A,g%,i%,n) factor

- A_1 is the starting cash flow
- There is NO base amount associated with a geometric gradient
- The remaining cash flows are generated from the A_1 starting value

Geometric Gradient: Notes

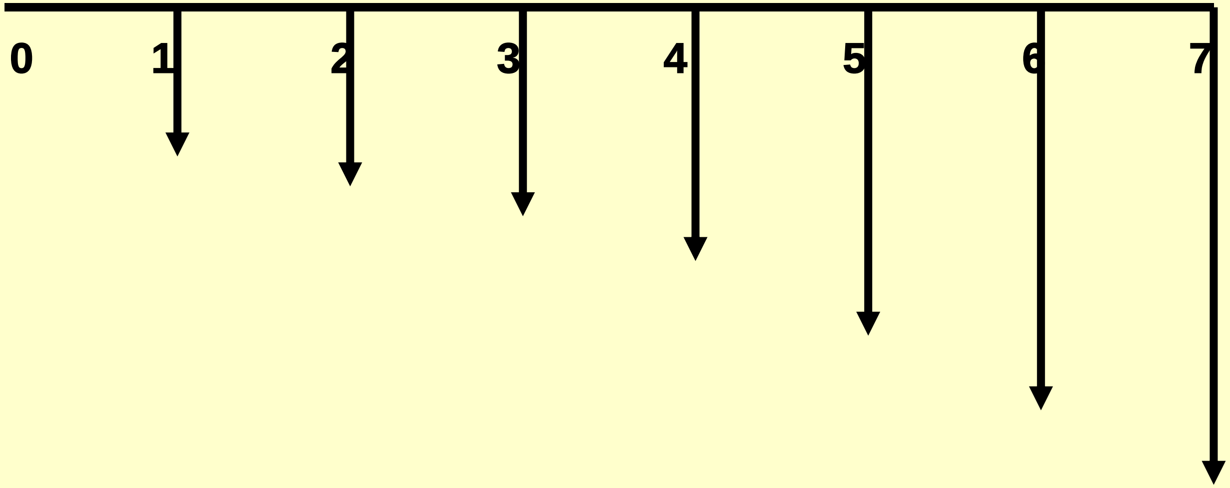
- One will not find tabulated tables for the (P/A, g,i,n) factor.
- You have to calculate either from the closed form for each problem or apply a pre-programmed spreadsheet model to find the needed factor value
- No spreadsheet built-in function for this factor!

Example 2.11

- **Assume maintenance costs for a particular activity will be $1700 one year from now.**
- **Assume an annual increase of 11% per year over a 6-year time period. (thus the total duration of study peroid is 7 yrs)**
- **If the interest rate is 8% per year, determine the present worth of the future expenses at time t = 0.**
- **First, draw a cash flow diagram to represent the model.**

Example 2.11

- g = +11% per period; A_1 = \$1700; i = 8%/yr



Example 2.11

- **P = $1,700(P/A,g=11%,i=8%,n=7)**
- **Need to calculate the P/A factor from the closed-form expression for a geometric gradient.**

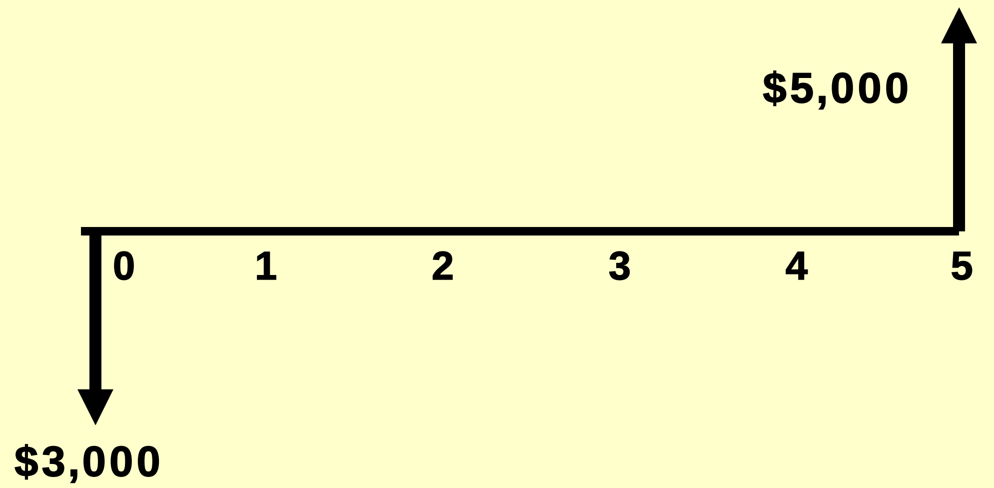
$$P_g = A_1\left[\frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}\right] \quad g \neq i$$

2.7 Determination of Unknown Interest Rate

- Class of problems where the interest rate, i%, is the unknown value
- **Termed, "rate of return analysis"(Chapter 7)**
- For simple, single payment problems (i.e., P and F only), solving for i% given the other parameters is not difficult
- For annuity and gradient type problems, solving for i% can be tedious
  - Trial and error method
  - Apply spreadsheet models

Example 2. 12

- Assume one can invest $3000 now in a venture in anticipation of gaining $5,000 in five (5) years.
- What interest rate equates these two cash flows?

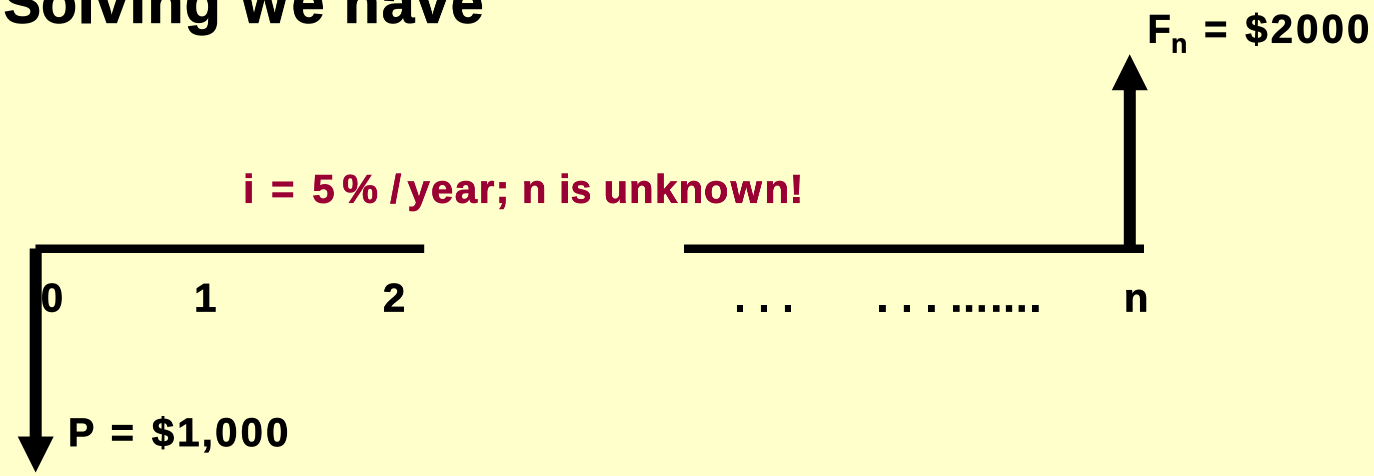


2.8 Determination of Unknown Number of Years

- **Some problems require knowing the number of time periods required given the other parameters**

Example 2. 14

- **How long will it take for $1,000 to double in value if the discount rate is 5% per year?**
- **Solving we have**



- $F_{n=?} = 1000(F/P,5\%,x)$: $2000 = 1000(1.05)^x$

Example 2. 14

- **Solving we have**