CHAPTER 11

11.1. Show that $E_{xs} = Ae^{jk_0 z+\phi}$ is a solution to the vector Helmholtz equation, Sec. 11.1, Eq. (16), for $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ and any ϕ and A: We take

$$\frac{d^2}{dz^2}\, Ae^{jk_0 z+\phi} = (jk_0)^2 Ae^{jk_0 z+\phi} = -k_0^2 E_{xs}$$

11.2. Let $\mathbf{E}(z,t) = 200 \sin 0.2z \cos 10^8 t\,\mathbf{a}_x + 500 \cos(0.2z + 50^\circ) \sin 10^8 t\,\mathbf{a}_y$ V/m. Find:

a) $\mathbf{E}$ at $P(0, 2, 0.6)$ at $t = 25$ ns: Obtain

$$\begin{aligned} E_P(t=25) &= 200 \sin[(0.2)(0.6)] \cos(2.5)\mathbf{a}_x + 500 \cos[(0.2)(0.6) + 50(2\pi)/360] \sin(2.5)\mathbf{a}_y \\ &= \underline{-19.2\mathbf{a}_x + 164\mathbf{a}_y \text{ V/m}} \end{aligned}$$

b) $|\mathbf{E}|$ at P at $t = 20$ ns:

$$\begin{aligned} E_P(t=20) &= 200 \sin[(0.2)(0.6)] \cos(2.0)\mathbf{a}_x + 500 \cos[(0.2)(0.6) + 50(2\pi)/360] \sin(2.0)\mathbf{a}_y \\ &= -9.96\mathbf{a}_x + 248\mathbf{a}_y \text{ V/m} \end{aligned}$$

Thus $|\mathbf{E}_P| = \sqrt{(9.96)^2 + (248)^2} = \underline{249 \text{ V/m}}$.

c) E_s at P: $E_s = 200 \sin 0.2z\mathbf{a}_x - j500 \cos(0.2z + 50^\circ)\mathbf{a}_y$. Thus

$$\begin{aligned} E_{sP} &= 200 \sin[(0.2)(0.6)]\,\mathbf{a}_x - j500 \cos[(0.2)(0.6) + 2\pi(50)/360]\,\mathbf{a}_y \\ &= \underline{23.9\mathbf{a}_x - j273\mathbf{a}_y \text{ V/m}} \end{aligned}$$

11.3. An $\mathbf{H}$ field in free space is given as $\mathbf{H}(x,t) = 10 \cos(10^8 t - \beta x)\mathbf{a}_y$ A/m. Find

a) β: Since we have a uniform plane wave, $\beta = \omega/c$, where we identify $\omega = 10^8$ sec^{-1}. Thus $\beta = 10^8/(3 \times 10^8) = \underline{0.33 \text{ rad/m}}$.

b) λ: We know $\lambda = 2\pi/\beta = \underline{18.9 \text{ m}}$.

c) $\mathbf{E}(x,t)$ at $P(0.1, 0.2, 0.3)$ at $t = 1$ ns: Use $E(x,t) = -\eta_0 H(x,t) = -(377)(10) \cos(10^8 t - \beta x) = -3.77 \times 10^3 \cos(10^8 t - \beta x)$. The vector direction of $\mathbf{E}$ will be $-\mathbf{a}_z$, since we require that $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, where $\mathbf{S}$ is x-directed. At the given point, the relevant coordinate is $x = 0.1$. Using this, along with $t = 10^{-9}$ sec, we finally obtain

$$\begin{aligned} \mathbf{E}(x,t) &= -3.77 \times 10^3 \cos[(10^8)(10^{-9}) - (0.33)(0.1)]\mathbf{a}_z = -3.77 \times 10^3 \cos(6.7 \times 10^{-2})\mathbf{a}_z \\ &= \underline{-3.76 \times 10^3 \mathbf{a}_z \text{ V/m}} \end{aligned}$$

11.4. In phasor form, the electric field intensity of a uniform plane wave in free space is expressed as $\mathbf{E}_s = (40 - j30)e^{-j20z}\mathbf{a}_x$ V/m. Find:

a) ω: From the given expression, we identify $\beta = 20$ rad/m. Then $\omega = c\beta = (3 \times 10^8)(20) = \underline{6.0 \times 10^9 \text{ rad/s}}$.

b) $\beta = \underline{20 \text{ rad/m}}$ from part a.

11.4. (continued)

c) $f = \omega/2\pi = \underline{956 \text{ MHz}}$.

d) $\lambda = 2\pi/\beta = 2\pi/20 = \underline{0.314 \text{ m}}$.

e) $\mathbf{H}_s$: In free space, we find $\mathbf{H}_s$ by dividing $\mathbf{E}_s$ by η_0, and assigning vector components such that $\mathbf{E}_s \times \mathbf{H}_s$ gives the required direction of wave travel: We find

$$\mathbf{H}_s = \frac{40 - j30}{377} e^{-j20z} \mathbf{a}_y = \underline{(0.11 - j0.08)e^{-j20z} \, \mathbf{a}_y \text{ A/m}}$$

f) $\mathbf{H}(z, t)$ at $P(6, -1, 0.07)$, $t = 71$ ps:

$$\mathbf{H}(z, t) = \text{Re}\left[\mathbf{H}_s e^{j\omega t}\right] = \left[0.11 \cos(6.0 \times 10^9 t - 20z) + 0.08 \sin(6.0 \times 10^9 t - 20z)\right] \mathbf{a}_y$$

Then

$$\begin{aligned}\mathbf{H}(.07, t = 71\text{ps}) &= \Big[0.11 \cos\left[(6.0 \times 10^9)(7.1 \times 10^{-11}) - 20(.07)\right] \\ &\quad + .08 \sin\left[(6.0 \times 10^9)(7.1 \times 10^{-11}) - 20(.07)\right]\Big] \mathbf{a}_y \\ &= [0.11(0.562) - 0.08(0.827)]\mathbf{a}_y = \underline{-6.2 \times 10^{-3} \mathbf{a}_y \text{ A/m}}\end{aligned}$$

11.5. A 150-MHz uniform plane wave in free space is described by $\mathbf{H}_s = (4 + j10)(2\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$ A/m.

a) Find numerical values for ω, λ, and β: First, $\omega = 2\pi \times 150 \times 10^6 = \underline{3\pi \times 10^8 \text{ sec}^{-1}}$. Second, for a uniform plane wave in free space, $\lambda = 2\pi c/\omega = c/f = (3 \times 10^8)/(1.5 \times 10^8) = \underline{2 \text{ m}}$. Third, $\beta = 2\pi/\lambda = \underline{\pi \text{ rad/m}}$.

b) Find $\mathbf{H}(z, t)$ at $t = 1.5$ ns, $z = 20$ cm: Use

$$\begin{aligned}\mathbf{H}(z, t) &= \text{Re}\{\mathbf{H}_s e^{j\omega t}\} = \text{Re}\{(4 + j10)(2\mathbf{a}_x + j\mathbf{a}_y)(\cos(\omega t - \beta z) + j \sin(\omega t - \beta z)\} \\ &= [8 \cos(\omega t - \beta z) - 20 \sin(\omega t - \beta z)] \, \mathbf{a}_x - [10 \cos(\omega t - \beta z) + 4 \sin(\omega t - \beta z)] \, \mathbf{a}_y\end{aligned}$$

. Now at the given position and time, $\omega t - \beta z = (3\pi \times 10^8)(1.5 \times 10^{-9}) - \pi(0.20) = \pi/4$. And $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$. So finally,

$$\mathbf{H}(z = 20\text{cm}, t = 1.5\text{ns}) = -\frac{1}{\sqrt{2}}\left(12\mathbf{a}_x + 14\mathbf{a}_y\right) = \underline{-8.5\mathbf{a}_x - 9.9\mathbf{a}_y \text{ A/m}}$$

c) What is $|E|_{max}$? Have $|E|_{max} = \eta_0 |H|_{max}$, where

$$|H|_{max} = \sqrt{\mathbf{H}_s \cdot \mathbf{H}_s^*} = [4(4 + j10)(4 - j10) + (j)(-j)(4 + j10)(4 - j10)]^{1/2} = 24.1 \text{ A/m}$$

Then $|E|_{max} = 377(24.1) = \underline{9.08 \text{ kV/m}}$.

11.6. Let $\mu_R = \epsilon_R = 1$ for the field $\mathbf{E}(z,t) = (25\mathbf{a}_x - 30\mathbf{a}_y)\cos(\omega t - 50z)$ V/m.

a) Find ω: $\omega = c\beta = (3 \times 10^8)(50) = \underline{15.0 \times 10^9 \text{ s}^{-1}}$.

b) Determine the displacement current density, $\mathbf{J}_d(z,t)$:

$$\mathbf{J}_d(z,t) = \frac{\partial \mathbf{D}}{\partial t} = -\epsilon_0 \omega (25\mathbf{a}_x - 30\mathbf{a}_y)\sin(\omega t - 50z)$$
$$= \underline{(-3.32\mathbf{a}_x + 3.98\mathbf{a}_y)\sin(1.5 \times 10^{10} t - 50z) \text{ A/m}^2}$$

c) Find the total magnetic flux Φ passing through the rectangle defined by $0 < x < 1$, $y = 0$, $0 < z < 1$, at $t = 0$: In free space, the magnetic field of the uniform plane wave can be easily found using the intrinsic impedance:

$$\mathbf{H}(z,t) = \left(\frac{25}{\eta_0}\mathbf{a}_y + \frac{30}{\eta_0}\mathbf{a}_x\right)\cos(\omega t - 50z) \text{ A/m}$$

Then $\mathbf{B}(z,t) = \mu_0 \mathbf{H}(z,t) = (1/c)(25\mathbf{a}_y + 30\mathbf{a}_x)\cos(\omega t - 50z)$ Wb/m^2, where $\mu_0/\eta_0 = \sqrt{\mu_0\epsilon_0} = 1/c$. The flux at $t = 0$ is now

$$\Phi = \int_0^1 \int_0^1 \mathbf{B} \cdot \mathbf{a}_y \, dx \, dz = \int_0^1 \frac{25}{c}\cos(50z)\, dz = \frac{25}{50(3 \times 10^8)}\sin(50) = \underline{-0.44 \text{ nWb}}$$

11.7. The phasor magnetic field intensity for a 400-MHz uniform plane wave propagating in a certain lossless material is $(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}$ A/m. Knowing that the maximum amplitude of $\mathbf{E}$ is 1500 V/m, find β, η, λ, v_p, ϵ_R, μ_R, and $\mathbf{H}(x,y,z,t)$: First, from the phasor expression, we identify $\beta = \underline{25 \text{ m}^{-1}}$ from the argument of the exponential function. Next, we evaluate $H_0 = |\mathbf{H}| = \sqrt{\mathbf{H} \cdot \mathbf{H}^*} = \sqrt{2^2 + 5^2} = \sqrt{29}$. Then $\eta = E_0/H_0 = 1500/\sqrt{29} = \underline{278.5\ \Omega}$. Then $\lambda = 2\pi/\beta = 2\pi/25 = .25 \text{ m} = \underline{25 \text{ cm}}$. Next,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 400 \times 10^6}{25} = \underline{1.01 \times 10^8 \text{ m/s}}$$

Now we note that

$$\eta = 278.5 = 377\sqrt{\frac{\mu_R}{\epsilon_R}} \quad \Rightarrow \quad \frac{\mu_R}{\epsilon_R} = 0.546$$

And

$$v_p = 1.01 \times 10^8 = \frac{c}{\sqrt{\mu_R \epsilon_R}} \quad \Rightarrow \quad \mu_R \epsilon_R = 8.79$$

We solve the above two equations simultaneously to find $\epsilon_R = \underline{4.01}$ and $\mu_R = \underline{2.19}$. Finally,

$$\mathbf{H}(x,y,z,t) = Re\left\{(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}e^{j\omega t}\right\}$$
$$= 2\cos(2\pi \times 400 \times 10^6 t - 25x)\mathbf{a}_y + 5\sin(2\pi \times 400 \times 10^6 t - 25x)\mathbf{a}_z$$
$$= \underline{2\cos(8\pi \times 10^8 t - 25x)\mathbf{a}_y + 5\sin(8\pi \times 10^8 t - 25x)\mathbf{a}_z \text{ A/m}}$$

11.8. Let the fields, $\mathbf{E}(z,t) = 1800\cos(10^7\pi t - \beta z)\mathbf{a}_x$ V/m and $\mathbf{H}(z,t) = 3.8\cos(10^7\pi t - \beta z)\mathbf{a}_y$ A/m, represent a uniform plane wave propagating at a velocity of 1.4×10^8 m/s in a perfect dielectric. Find:

a) $\beta = \omega/v = (10^7\pi)/(1.4 \times 10^8) = \underline{0.224\,\text{m}^{-1}}$.

b) $\lambda = 2\pi/\beta = 2\pi/.224 = \underline{28.0\,\text{m}}$.

c) $\eta = |\mathbf{E}|/|\mathbf{H}| = 1800/3.8 = \underline{474\,\Omega}$.

d) μ_R: Have two equations in the two unknowns, μ_R and ϵ_R: $\eta = \eta_0\sqrt{\mu_R/\epsilon_R}$ and $\beta = \omega\sqrt{\mu_R\epsilon_R}/c$. Eliminate ϵ_R to find

$$\mu_R = \left[\frac{\beta c \eta}{\omega \eta_0}\right]^2 = \left[\frac{(.224)(3 \times 10^8)(474)}{(10^7\pi)(377)}\right]^2 = \underline{2.69}$$

e) $\epsilon_R = \mu_R(\eta_0/\eta)^2 = (2.69)(377/474)^2 = \underline{1.70}$.

11.9. A certain lossless material has $\mu_R = 4$ and $\epsilon_R = 9$. A 10-MHz uniform plane wave is propagating in the $\mathbf{a}_y$ direction with $E_{x0} = 400$ V/m and $E_{y0} = E_{z0} = 0$ at $P(0.6, 0.6, 0.6)$ at $t = 60$ ns.

a) Find β, λ, v_p, and η: For a uniform plane wave,

$$\beta = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}\sqrt{\mu_R\epsilon_R} = \frac{2\pi \times 10^7}{3 \times 10^8}\sqrt{(4)(9)} = \underline{0.4\pi \text{ rad/m}}$$

Then $\lambda = (2\pi)/\beta = (2\pi)/(0.4\pi) = \underline{5\,\text{m}}$. Next,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^7}{4\pi \times 10^{-1}} = \underline{5 \times 10^7 \text{ m/s}}$$

Finally,

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \eta_0\sqrt{\frac{\mu_R}{\epsilon_R}} = 377\sqrt{\frac{4}{9}} = \underline{251\ \Omega}$$

b) Find $E(t)$ (at P): We are given the amplitude at $t = 60$ ns and at $y = 0.6$ m. Let the maximum amplitude be E_{max}, so that in general, $E_x = E_{max}\cos(\omega t - \beta y)$. At the given position and time,

$$E_x = 400 = E_{max}\cos[(2\pi \times 10^7)(60 \times 10^{-9}) - (4\pi \times 10^{-1})(0.6)] = E_{max}\cos(0.96\pi)$$
$$= -0.99E_{max}$$

So $E_{max} = (400)/(-0.99) = -403$ V/m. Thus at P, $E(t) = \underline{-403\cos(2\pi \times 10^7 t) \text{ V/m}}$.

c) Find $H(t)$: First, we note that if E at a given instant points in the negative x direction, while the wave propagates in the forward y direction, then H at that same position and time must point in the positive z direction. Since we have a lossless homogeneous medium, η is real, and we are allowed to write $H(t) = E(t)/\eta$, where η is treated as negative and real. Thus

$$H(t) = H_z(t) = \frac{E_x(t)}{\eta} = \frac{-403}{-251}\cos(2\pi \times 10^{-7}t) = \underline{1.61\cos(2\pi \times 10^{-7}t) \text{ A/m}}$$

11.10. Given a 20MHz uniform plane wave with $\mathbf{H}_s = (6\mathbf{a}_x - j2\mathbf{a}_y)e^{-jz}$ A/m, assume propagation in a lossless medium characterized by $\epsilon_R = 5$ and an unknown μ_R.

a) Find λ, v_p, μ_R, and η: First, $\beta = 1$, so $\lambda = 2\pi/\beta = \underline{2\pi \text{ m}}$. Next, $v_p = \omega/\beta = 2\pi \times 20 \times 10^6 = \underline{4\pi \times 10^7 \text{ m/s}}$. Then, $\mu_R = (\beta^2 c^2)/(\omega^2 \epsilon_R) = (3 \times 10^8)^2/(4\pi \times 10^7)^2(5) = \underline{1.14}$. Finally, $\eta = \eta_0\sqrt{\mu_R/\epsilon_R} = 377\sqrt{1.14/5} = \underline{180}$.

b) Determine $\mathbf{E}$ at the origin at $t = 20$ns: We use the relation $|\mathbf{E}| = \eta|\mathbf{H}|$ and note that for positive z propagation, a positive x component of $\mathbf{H}$ is coupled to a negative y component of $\mathbf{E}$, and a negative y component of $\mathbf{H}$ is coupled to a negative x component of $\mathbf{E}$. We obtain $\mathbf{E}_s = -\eta(6\mathbf{a}_y + j2\mathbf{a}_x)e^{-jz}$. Then $\mathbf{E}(z,t) = \text{Re}\left\{\mathbf{E}_s e^{j\omega t}\right\} = -6\eta\cos(\omega t - z)\mathbf{a}_y + 2\eta\sin(\omega t - z)\mathbf{a}_x = 360\sin(\omega t - z)\mathbf{a}_x - 1080\cos(\omega t - z)\mathbf{a}_y$. With $\omega = 4\pi \times 10^7$ sec^{-1}, $t = 2 \times 10^{-8}$ s, and $z = 0$, $\mathbf{E}$ evaluates as $\mathbf{E}(0, 20\text{ns}) = 360(0.588)\mathbf{a}_x - 1080(-0.809)\mathbf{a}_y = \underline{212\mathbf{a}_x + 874\mathbf{a}_y \text{ V/m}}$.

11.11. A 2-GHz uniform plane wave has an amplitude of $E_{y0} = 1.4$ kV/m at $(0, 0, 0, t = 0)$ and is propagating in the $\mathbf{a}_z$ direction in a medium where $\epsilon'' = 1.6 \times 10^{-11}$ F/m, $\epsilon' = 3.0 \times 10^{-11}$ F/m, and $\mu = 2.5\,\mu$H/m. Find:

a) E_y at $P(0, 0, 1.8\text{cm})$ at 0.2 ns: To begin, we have the ratio, $\epsilon''/\epsilon' = 1.6/3.0 = 0.533$. So

$$\alpha = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]^{1/2}$$

$$= (2\pi \times 2 \times 10^9)\sqrt{\frac{(2.5 \times 10^{-6})(3.0 \times 10^{-11})}{2}}\left[\sqrt{1 + (.533)^2} - 1\right]^{1/2} = 28.1\,\text{Np/m}$$

Then

$$\beta = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]^{1/2} = 112\,\text{rad/m}$$

Thus in general,

$$E_y(z,t) = 1.4e^{-28.1z}\cos(4\pi \times 10^9 t - 112z)\ \text{kV/m}$$

Evaluating this at $t = 0.2$ ns and $z = 1.8$ cm, find

$$E_y(1.8\,\text{cm}, 0.2\,\text{ns}) = \underline{0.74\ \text{kV/m}}$$

b) H_x at P at 0.2 ns: We use the phasor relation, $H_{xs} = -E_{ys}/\eta$ where

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} = \sqrt{\frac{2.5 \times 10^{-6}}{3.0 \times 10^{-11}}}\frac{1}{\sqrt{1 - j(.533)}} = 263 + j65.7 = 271\angle\,14^\circ\ \Omega$$

So now

$$H_{xs} = -\frac{E_{ys}}{\eta} = -\frac{(1.4 \times 10^3)e^{-28.1z}e^{-j112z}}{271e^{j14^\circ}} = -5.16e^{-28.1z}e^{-j112z}e^{-j14^\circ}\ \text{A/m}$$

Then

$$H_x(z,t) = -5.16e^{-28.1z}\cos(4\pi \times 10^{-9}t - 112z - 14^\circ)$$

This, when evaluated at $t = 0.2$ ns and $z = 1.8$ cm, yields

$$H_x(1.8\,\text{cm}, 0.2\,\text{ns}) = \underline{-3.0\,\text{A/m}}$$

11.12. The plane wave $\mathbf{E}_s = 300e^{-jkx}\mathbf{a}_y$ V/m is propagating in a material for which $\mu = 2.25\ \mu$H/m, $\epsilon' = 9$ pF/m, and $\epsilon'' = 7.8$ pF/m. If $\omega = 64$ Mrad/s, find:

a) α: We use the general formula, Eq. (35):

$$\alpha = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}-1\right]^{1/2}$$
$$= (64\times10^6)\sqrt{\frac{(2.25\times10^{-6})(9\times10^{-12})}{2}}\left[\sqrt{1+(.867)^2}-1\right]^{1/2} = \underline{0.116\text{ Np/m}}$$

b) β: Using (36), we write

$$\beta = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}+1\right]^{1/2} = \underline{.311\text{ rad/m}}$$

c) $v_p = \omega/\beta = (64\times10^6)/(.311) = \underline{2.06\times10^8\text{ m/s}}$.

d) $\lambda = 2\pi/\beta = 2\pi/(.311) = \underline{20.2\text{ m}}$.

e) η: Using (39):

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1-j(\epsilon''/\epsilon')}} = \sqrt{\frac{2.25\times10^{-6}}{9\times10^{-12}}}\frac{1}{\sqrt{1-j(.867)}} = 407+j152 = \underline{434.5e^{j.36}\ \Omega}$$

f) $\mathbf{H}_s$: With $\mathbf{E}_s$ in the positive y direction (at a given time) and propagating in the positive x direction, we would have a positive z component of $\mathbf{H}_s$, at the same time. We write (with $jk = \alpha + j\beta$):

$$\mathbf{H}_s = \frac{E_s}{\eta}\mathbf{a}_z = \frac{300}{434.5e^{j.36}}e^{-jkx}\mathbf{a}_z = 0.69e^{-\alpha x}e^{-j\beta x}e^{-j.36}\mathbf{a}_z$$
$$= \underline{0.69e^{-.116x}e^{-j.311x}e^{-j.36}\mathbf{a}_z\text{ A/m}}$$

g) $\mathbf{E}(3, 2, 4, 10\text{ns})$: The real instantaneous form of $\mathbf{E}$ will be

$$\mathbf{E}(x,y,z,t) = \text{Re}\left\{\mathbf{E}_s e^{j\omega t}\right\} = 300e^{-\alpha x}\cos(\omega t - \beta x)\mathbf{a}_y$$

Therefore

$$\mathbf{E}(3, 2, 4, 10\text{ns}) = 300e^{-.116(3)}\cos[(64\times10^6)(10^{-8}) - .311(3)]\mathbf{a}_y = \underline{203\text{ V/m}}$$

11.13. Let $jk = 0.2 + j1.5\,\text{m}^{-1}$ and $\eta = 450 + j60\ \Omega$ for a uniform plane wave propagating in the $\mathbf{a}_z$ direction. If $\omega = 300$ Mrad/s, find μ, ϵ', and ϵ'': We begin with

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1-j(\epsilon''/\epsilon')}} = 450 + j60$$

and

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1-j(\epsilon''/\epsilon')} = 0.2 + j1.5$$

11.13. (continued) Then

$$\eta\eta^* = \frac{\mu}{\epsilon'} \frac{1}{\sqrt{1+(\epsilon''/\epsilon')^2}} = (450 + j60)(450 - j60) = 2.06 \times 10^5 \qquad (1)$$

and

$$(jk)(jk)^* = \omega^2 \mu\epsilon' \sqrt{1+(\epsilon''/\epsilon')^2} = (0.2 + j1.5)(0.2 - j1.5) = 2.29 \qquad (2)$$

Taking the ratio of (2) to (1),

$$\frac{(jk)(jk)^*}{\eta\eta^*} = \omega^2(\epsilon')^2 \left(1 + (\epsilon''/\epsilon')^2\right) = \frac{2.29}{2.06 \times 10^5} = 1.11 \times 10^{-5}$$

Then with $\omega = 3 \times 10^8$,

$$(\epsilon')^2 = \frac{1.11 \times 10^{-5}}{(3 \times 10^8)^2 \left(1 + (\epsilon''/\epsilon')^2\right)} = \frac{1.23 \times 10^{-22}}{\left(1 + (\epsilon''/\epsilon')^2\right)} \qquad (3)$$

Now, we use Eqs. (35) and (36). Squaring these and taking their ratio gives

$$\frac{\alpha^2}{\beta^2} = \frac{\sqrt{1+(\epsilon''/\epsilon')^2}}{\sqrt{1+(\epsilon''/\epsilon')^2}} = \frac{(0.2)^2}{(1.5)^2}$$

We solve this to find $\epsilon''/\epsilon' = 0.271$. Substituting this result into (3) gives $\epsilon' = 1.07 \times 10^{-11}$ F/m. Since $\epsilon''/\epsilon' = 0.271$, we then find $\epsilon'' = 2.90 \times 10^{-12}$ F/m. Finally, using these results in either (1) or (2) we find $\mu = 2.28 \times 10^{-6}$ H/m. Summary: $\mu = \underline{2.28 \times 10^{-6}\text{ H/m}}$,
$\epsilon' = \underline{1.07 \times 10^{-11}\text{ F/m}}$, and $\epsilon'' = \underline{2.90 \times 10^{-12}\text{ F/m}}$.

11.14. A certain nonmagnetic material has the material constants $\epsilon'_R = 2$ and $\epsilon''/\epsilon' = 4 \times 10^{-4}$ at $\omega = 1.5$ Grad/s. Find the distance a uniform plane wave can propagate through the material before:

a) it is attenuated by 1 Np: First, $\epsilon'' = (4 \times 10^4)(2)(8.854 \times 10^{-12}) = 7.1 \times 10^{-15}$ F/m. Then, since $\epsilon''/\epsilon' << 1$, we use the approximate form for α, given by Eq. (51) (written in terms of ϵ''):

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{(1.5 \times 10^9)(7.1 \times 10^{-15})}{2} \frac{377}{\sqrt{2}} = 1.42 \times 10^{-3}\text{ Np/m}$$

The required distance is now $z_1 = (1.42 \times 10^{-3})^{-1} = \underline{706\text{ m}}$

b) the power level is reduced by one-half: The governing relation is $e^{-2\alpha z_{1/2}} = 1/2$, or $z_{1/2} = \ln 2/2\alpha = \ln 2/2(1.42 \times 10^-3) = \underline{244\text{ m}}$.

c) the phase shifts 360°: This distance is defined as one wavelength, where $\lambda = 2\pi/\beta$ $= (2\pi c)/(\omega\sqrt{\epsilon'_R}) = [2\pi(3 \times 10^8)]/[(1.5 \times 10^9)\sqrt{2}] = \underline{0.89\text{ m}}$.

11.15. A 10 GHz radar signal may be represented as a uniform plane wave in a sufficiently small region. Calculate the wavelength in centimeters and the attenuation in nepers per meter if the wave is propagating in a non-magnetic material for which

a) $\epsilon'_R = 1$ and $\epsilon''_R = 0$: In a non-magnetic material, we would have:

$$\alpha = \omega\sqrt{\frac{\mu_0\epsilon_0\epsilon'_R}{2}} \left[\sqrt{1 + \left(\frac{\epsilon''_R}{\epsilon'_R}\right)^2} - 1\right]^{1/2}$$

11.15. (continued) and

$$\beta = \omega\sqrt{\frac{\mu_0\epsilon_0\epsilon'_R}{2}}\left[\sqrt{1+\left(\frac{\epsilon''_R}{\epsilon'_R}\right)^2}+1\right]^{1/2}$$

With the given values of ϵ'_R and ϵ''_R, it is clear that $\beta = \omega\sqrt{\mu_0\epsilon_0} = \omega/c$, and so $\lambda = 2\pi/\beta = 2\pi c/\omega = 3\times 10^{10}/10^{10} = \underline{3\text{ cm}}$. It is also clear that $\alpha = \underline{0}$.

b) $\epsilon'_R = 1.04$ and $\epsilon''_R = 9.00\times 10^{-4}$: In this case $\epsilon''_R/\epsilon'_R << 1$, and so $\beta \doteq \omega\sqrt{\epsilon'_R}/c = 2.13\text{ cm}^{-1}$. Thus $\lambda = 2\pi/\beta = \underline{2.95\text{ cm}}$. Then

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\omega\epsilon''_R}{2}\frac{\sqrt{\mu_0\epsilon_0}}{\sqrt{\epsilon'_R}} = \frac{\omega}{2c}\frac{\epsilon''_R}{\sqrt{\epsilon'_R}} = \frac{2\pi\times 10^{10}}{2\times 3\times 10^8}\frac{(9.00\times 10^{-4})}{\sqrt{1.04}}$$
$$= \underline{9.24\times 10^{-2}\text{ Np/m}}$$

c) $\epsilon'_R = 2.5$ and $\epsilon''_R = 7.2$: Using the above formulas, we obtain

$$\beta = \frac{2\pi\times 10^{10}\sqrt{2.5}}{(3\times 10^{10})\sqrt{2}}\left[\sqrt{1+\left(\frac{7.2}{2.5}\right)^2}+1\right]^{1/2} = 4.71\text{ cm}^{-1}$$

and so $\lambda = 2\pi/\beta = \underline{1.33\text{ cm}}$. Then

$$\alpha = \frac{2\pi\times 10^{10}\sqrt{2.5}}{(3\times 10^{8})\sqrt{2}}\left[\sqrt{1+\left(\frac{7.2}{2.5}\right)^2}-1\right]^{1/2} = \underline{335\text{ Np/m}}$$

11.16. The power factor of a capacitor is defined as the cosine of the impedance phase angle, and its Q is ωCR, where R is the parallel resistance. Assume an idealized parallel plate capacitor having a dielecric characterized by σ, ϵ', and μ_R. Find both the power factor and Q in terms of the loss tangent: First, the impedance will be:

$$Z = \frac{R\left(\frac{1}{j\omega C}\right)}{R+\left(\frac{1}{j\omega C}\right)} = R\frac{1-jR\omega C}{1+(R\omega C)^2} = R\frac{1-jQ}{1+Q^2}$$

Now $R = d/(\sigma A)$ and $C = \epsilon' A/d$, and so $Q = \omega\epsilon'/\sigma = \underline{1/l.t.}$ Then the power factor is P.F $= \cos[\tan^{-1}(-Q)] = \underline{1/\sqrt{1+Q^2}}$.

11.17. Let $\eta = 250 + j30\,\Omega$ and $jk = 0.2 + j2\,\mathrm{m}^{-1}$ for a uniform plane wave propagating in the $\mathbf{a}_z$ direction in a dielectric having some finite conductivity. If $|E_s| = 400$ V/m at $z = 0$, find:

a) $\mathbf{P}_{z,av}$ at $z = 0$ and $z = 60$ cm: Assume x-polarization for the electric field. Then

$$\mathbf{P}_{z,av} = \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\mathrm{Re}\left\{400e^{-\alpha z}e^{-j\beta z}\mathbf{a}_x \times \frac{400}{\eta^*}e^{-\alpha z}e^{j\beta z}\mathbf{a}_y\right\}$$
$$= \frac{1}{2}(400)^2 e^{-2\alpha z}\mathrm{Re}\left\{\frac{1}{\eta^*}\right\}\mathbf{a}_z = 8.0 \times 10^4 e^{-2(0.2)z}\mathrm{Re}\left\{\frac{1}{250 - j30}\right\}\mathbf{a}_z$$
$$= 315\, e^{-2(0.2)z}\,\mathbf{a}_z\ \mathrm{W/m}^2$$

Evaluating at $z = 0$, obtain $\mathbf{P}_{z,av}(z = 0) = \underline{315\,\mathbf{a}_z\ \mathrm{W/m}^2}$,
and at $z = 60$ cm, $\mathbf{P}_{z,av}(z = 0.6) = 315e^{-2(0.2)(0.6)}\mathbf{a}_z = \underline{248\,\mathbf{a}_z\ \mathrm{W/m}^2}$.

b) the average ohmic power dissipation in watts per cubic meter at $z = 60$ cm: At this point a flaw becomes evident in the problem statement, since solving this part in two different ways gives results that are not the same. I will demonstrate: In the first method, we use Poynting's theorem in point form (first equation at the top of p. 366), which we modify for the case of time-average fields to read:

$$-\nabla \cdot \mathbf{P}_{z,av} = <\mathbf{J}\cdot\mathbf{E}>$$

where the right hand side is the average power dissipation per volume. Note that the additional right-hand-side terms in Poynting's theorem that describe changes in energy stored in the fields will both be zero in steady state. We apply our equation to the result of part a:

$$<\mathbf{J}\cdot\mathbf{E}> = -\nabla\cdot\mathbf{P}_{z,av} = -\frac{d}{dz}315\,e^{-2(0.2)z} = (0.4)(315)e^{-2(0.2)z} = 126e^{-0.4z}\ \mathrm{W/m}^3$$

At $z = 60$ cm, this becomes $<\mathbf{J}\cdot\mathbf{E}> = 99.1\ \mathrm{W/m}^3$. In the second method, we solve for the conductivity and evaluate $<\mathbf{J}\cdot\mathbf{E}> = \sigma <E^2>$. We use

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j(\epsilon''/\epsilon')}$$

and

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}}$$

We take the ratio,

$$\frac{jk}{\eta} = j\omega\epsilon'\left[1 - j\left(\frac{\epsilon''}{\epsilon'}\right)\right] = j\omega\epsilon' + \omega\epsilon''$$

Identifying $\sigma = \omega\epsilon''$, we find

$$\sigma = \mathrm{Re}\left\{\frac{jk}{\eta}\right\} = \mathrm{Re}\left\{\frac{0.2 + j2}{250 + j30}\right\} = 1.74 \times 10^{-3}\ \mathrm{S/m}$$

Now we find the dissipated power per volume:

$$\sigma <E^2> = 1.74 \times 10^{-3}\left(\frac{1}{2}\right)\left(400e^{-0.2z}\right)^2$$

11.17b. (continued) At $z = 60$ cm, this evaluates as 109 W/m^3. One can show that consistency between the two methods requires that

$$\text{Re}\left\{\frac{1}{\eta^*}\right\} = \frac{\sigma}{2\alpha}$$

This relation does not hold using the numbers as given in the problem statement and the value of σ found above. Note that in Problem 11.13, where all values are worked out, the relation does hold and consistent results are obtained using both methods.

11.18a. Find $P(\mathbf{r}, t)$ if $\mathbf{E}_s = 400e^{-j2x}\mathbf{a}_y$ V/m in free space: A positive y component of $\mathbf{E}$ requires a positive z component of $\mathbf{H}$ for propagation in the forward x direction. Thus $\mathbf{H}_s = (400/\eta_0)e^{-j2x}\mathbf{a}_z = 1.06e^{-j2x}\mathbf{a}_z$ A/m. In real form, the field are $\mathbf{E}(x, t) = 400\cos(\omega t - 2x)\mathbf{a}_y$ and $\mathbf{H}(x, t) = 1.06\cos(\omega t - 2x)\mathbf{a}_z$. Now $P(\mathbf{r}, t) = P(x, t) = \mathbf{E}(x, t) \times \mathbf{H}(x, t) = \underline{424.4\cos^2(\omega t - 2x)\mathbf{a}_x \text{ W/m}^2}$.

b) Find P at $t = 0$ for $\mathbf{r} = (a, 5, 10)$, where a = 0,1,2, and 3: At $t = 0$, we find from part a, $P(a, 0) = 424.4\cos^2(2a)$, which leads to the values (in W/m^2): $\underline{424.4 \text{ at } a = 0}$, $\underline{73.5 \text{ at } a = 1}$, $\underline{181.3 \text{ at } a = 2}$, and $\underline{391.3 \text{ at } a = 3}$.

c) Find P at the origin for $T = 0$, $0.2T$, $0.4T$, and $0.6T$, where T is the oscillation period. At the origin, we have $P(0, t) = 424.4\cos^2(\omega t) = 424.4\cos^2(2\pi t/T)$. Using this, we obtain the following values (in W/m^2): $\underline{424.4 \text{ at } t = 0}$, $\underline{42.4 \text{ at } t = 0.2T}$, $\underline{277.8 \text{ at } t = 0.4T}$, and $\underline{277.8 \text{ at } t = 0.6T}$.

11.19. Perfectly-conducting cylinders with radii of 8 mm and 20 mm are coaxial. The region between the cylinders is filled with a perfect dielectric for which $\epsilon = 10^{-9}/4\pi$ F/m and $\mu_R = 1$. If $\mathbf{E}$ in this region is $(500/\rho)\cos(\omega t - 4z)\mathbf{a}_\rho$ V/m, find:

a) ω, with the help of Maxwell's equations in cylindrical coordinates: We use the two curl equations, beginning with $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$, where in this case,

$$\nabla \times \mathbf{E} = \frac{\partial E_\rho}{\partial z}\mathbf{a}_\phi = \frac{2000}{\rho}\sin(\omega t - 4z)\mathbf{a}_\phi = -\frac{\partial B_\phi}{\partial t}\mathbf{a}_\phi$$

So

$$B_\phi = \int \frac{2000}{\rho}\sin(\omega t - 4z)dt = \frac{2000}{\omega\rho}\cos(\omega t - 4z) \text{ T}$$

Then

$$H_\phi = \frac{B_\phi}{\mu_0} = \frac{2000}{(4\pi \times 10^{-7})\omega\rho}\cos(\omega t - 4z) \text{ A/m}$$

We next use $\nabla \times \mathbf{H} = \partial\mathbf{D}/\partial t$, where in this case

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z$$

where the second term on the right hand side becomes zero when substituting our H_ϕ. So

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho = -\frac{8000}{(4\pi \times 10^{-7})\omega\rho}\sin(\omega t - 4z)\mathbf{a}_\rho = \frac{\partial D_\rho}{\partial t}\mathbf{a}_\rho$$

And

$$D_\rho = \int -\frac{8000}{(4\pi \times 10^{-7})\omega\rho}\sin(\omega t - 4z)dt = \frac{8000}{(4\pi \times 10^{-7})\omega^2\rho}\cos(\omega t - 4z) \text{ C/m}^2$$

11.19a. (continued) Finally, using the given ϵ,

$$E_\rho = \frac{D_\rho}{\epsilon} = \frac{8000}{(10^{-16})\omega^2\rho}\cos(\omega t - 4z) \text{ V/m}$$

This must be the same as the given field, so we require

$$\frac{8000}{(10^{-16})\omega^2\rho} = \frac{500}{\rho} \quad \Rightarrow \quad \omega = \underline{4 \times 10^8 \text{ rad/s}}$$

b) $\mathbf{H}(\rho, z, t)$: From part a, we have

$$\mathbf{H}(\rho, z, t) = \frac{2000}{(4\pi \times 10^{-7})\omega\rho}\cos(\omega t - 4z)\mathbf{a}_\phi = \underline{\frac{4.0}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\phi \text{ A/m}}$$

c) $\mathbf{P}(\rho, \phi, z)$: This will be

$$\mathbf{P}(\rho, \phi, z) = \mathbf{E} \times \mathbf{H} = \frac{500}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\rho \times \frac{4.0}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\phi$$
$$= \underline{\frac{2.0 \times 10^{-3}}{\rho^2}\cos^2(4 \times 10^8 t - 4z)\mathbf{a}_z \text{ W/m}^2}$$

d) the average power passing through every cross-section $8 < \rho < 20$ mm, $0 < \phi < 2\pi$. Using the result of part c, we find $\mathbf{P}_{avg} = (1.0 \times 10^3)/\rho^2 \mathbf{a}_z$ W/m^2. The power through the given cross-section is now

$$\text{P} = \int_0^{2\pi}\int_{.008}^{.020} \frac{1.0 \times 10^3}{\rho^2}\,\rho\,d\rho\,d\phi = 2\pi \times 10^3 \ln\left(\frac{20}{8}\right) = \underline{5.7 \text{ kW}}$$

11.20. If $\mathbf{E}_s = (60/r)\sin\theta\, e^{-j2r}\,\mathbf{a}_\theta$ V/m, and $\mathbf{H}_s = (1/4\pi r)\sin\theta\, e^{-j2r}\,\mathbf{a}_\phi$ A/m in free space, find the average power passing outward through the surface $r = 10^6$, $0 < \theta < \pi/3$, and $0 < \phi < 2\pi$.

$$P_{avg} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{15\sin^2\theta}{2\pi r^2}\,\mathbf{a}_r \text{ W/m}^2$$

Then, the requested power will be

$$\Phi = \int_0^{2\pi}\int_0^{\pi/3} \frac{15\sin^2\theta}{2\pi r^2}\,\mathbf{a}_r \cdot \mathbf{a}_r\, r^2 \sin\theta\, d\theta\, d\phi = 15\int_0^{\pi/3}\sin^3\theta\, d\theta$$
$$= 15\left(-\frac{1}{3}\cos\theta(\sin^2\theta + 2)\right)\Big|_0^{\pi/3} = \frac{25}{8} = \underline{3.13 \text{ W}}$$

Note that the radial distance at the surface, $r = 10^6$ m, makes no difference, since the power density dimishes as $1/r^2$.

11.21. The cylindrical shell, 1 cm < ρ < 1.2 cm, is composed of a conducting material for which $\sigma = 10^6$ S/m. The external and internal regions are non-conducting. Let $H_\phi = 2000$ A/m at $\rho = 1.2$ cm.

a) Find **H** everywhere: Use Ampere's circuital law, which states:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho(2000) = 2\pi(1.2 \times 10^{-2})(2000) = 48\pi \text{ A} = I_{encl}$$

Then in this case

$$\mathbf{J} = \frac{I}{Area}\mathbf{a}_z = \frac{48}{(1.44 - 1.00) \times 10^{-4}}\mathbf{a}_z = 1.09 \times 10^6 \,\mathbf{a}_z \text{ A/m}^2$$

With this result we again use Ampere's circuital law to find **H** everywhere within the shell as a function of ρ (in meters):

$$H_{\phi 1}(\rho) = \frac{1}{2\pi\rho}\int_0^{2\pi}\int_{.01}^{\rho} 1.09 \times 10^6 \;\rho\, d\rho\, d\phi = \underline{\frac{54.5}{\rho}(10^4\rho^2 - 1) \text{ A/m} \;\; (.01 < \rho < .012)}$$

Outside the shell, we would have

$$H_{\phi 2}(\rho) = \frac{48\pi}{2\pi\rho} = \underline{24/\rho \text{ A/m} \;\; (\rho > .012)}$$

Inside the shell ($\rho < .01$ m), $H_\phi = 0$ since there is no enclosed current.

b) Find **E** everywhere: We use

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{1.09 \times 10^6}{10^6}\mathbf{a}_z = \underline{1.09\,\mathbf{a}_z \text{ V/m}}$$

which is valid, presumeably, outside as well as inside the shell.

c) Find **P** everywhere: Use

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = 1.09\,\mathbf{a}_z \times \frac{54.5}{\rho}(10^4\rho^2 - 1)\,\mathbf{a}_\phi$$
$$= \underline{-\frac{59.4}{\rho}(10^4\rho^2 - 1)\,\mathbf{a}_\rho \text{ W/m}^2 \;\; (.01 < \rho < .012\,\text{m})}$$

Outside the shell,

$$\mathbf{P} = 1.09\,\mathbf{a}_z \times \frac{24}{\rho}\,\mathbf{a}_\phi = \underline{-\frac{26}{\rho}\,\mathbf{a}_\rho \text{ W/m}^2 \;\; (\rho > .012\,\text{m})}$$

11.22. The inner and outer dimensions of a copper coaxial transmission line are 2 and 7 mm, respectively. Both conductors have thicknesses much greater than δ. The dielectric is lossless and the operating frequency is 400 MHz. Calculate the resistance per meter length of the:

a) inner conductor: First

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(4 \times 10^8)(4\pi \times 10^{-7})(5.8 \times 10^7)}} = 3.3 \times 10^{-6}\text{m} = 3.3\mu\text{m}$$

Now, using (70) with a unit length, we find

$$R_{in} = \frac{1}{2\pi a \sigma \delta} = \frac{1}{2\pi(2 \times 10^{-3})(5.8 \times 10^7)(3.3 \times 10^{-6})} = \underline{0.42 \text{ ohms/m}}$$

b) outer conductor: Again, (70) applies but with a different conductor radius. Thus

$$R_{out} = \frac{a}{b} R_{in} = \frac{2}{7}(0.42) = \underline{0.12 \text{ ohms/m}}$$

c) transmission line: Since the two resistances found above are in series, the line resistance is their sum, or $R = R_{in} + R_{out} = \underline{0.54 \text{ ohms/m}}$.

11.23. A hollow tubular conductor is constructed from a type of brass having a conductivity of 1.2×10^7 S/m. The inner and outer radii are 9 mm and 10 mm respectively. Calculate the resistance per meter length at a frequency of

a) dc: In this case the current density is uniform over the entire tube cross-section. We write:

$$R(\text{dc}) = \frac{L}{\sigma A} = \frac{1}{(1.2 \times 10^7)\pi(.01^2 - .009^2)} = \underline{1.4 \times 10^{-3}\ \Omega/\text{m}}$$

b) 20 MHz: Now the skin effect will limit the effective cross-section. At 20 MHz, the skin depth is

$$\delta(20\text{MHz}) = [\pi f \mu_0 \sigma]^{-1/2} = [\pi(20 \times 10^6)(4\pi \times 10^{-7})(1.2 \times 10^7)]^{-1/2} = 3.25 \times 10^{-5}\text{ m}$$

This is much less than the outer radius of the tube. Therefore we can approximate the resistance using the formula:

$$R(20\text{MHz}) = \frac{L}{\sigma A} = \frac{1}{2\pi b \delta} = \frac{1}{(1.2 \times 10^7)(2\pi(.01))(3.25 \times 10^{-5})} = \underline{4.1 \times 10^{-2}\ \Omega/\text{m}}$$

c) 2 GHz: Using the same formula as in part b, we find the skin depth at 2 GHz to be $\delta = 3.25 \times 10^{-6}$ m. The resistance (using the other formula) is $R(2\text{GHz}) = \underline{4.1 \times 10^{-1}\ \Omega/\text{m}}$.

11.24a. Most microwave ovens operate at 2.45 GHz. Assume that $\sigma = 1.2 \times 10^6$ S/m and $\mu_R = 500$ for the stainless steel interior, and find the depth of penetration:

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(2.45 \times 10^9)(4\pi \times 10^{-7})(1.2 \times 10^6)}} = 9.28 \times 10^{-6}\text{m} = 9.28\mu\text{m}$$

b) Let $E_s = 50\angle\, 0°$ V/m at the surface of the conductor, and plot a curve of the amplitude of E_s vs. the angle of E_s as the field propagates into the stainless steel: Since the conductivity is high, we use (62) to write $\alpha \doteq \beta \doteq \sqrt{\pi f \mu \sigma} = 1/\delta$. So, assuming that the direction into the conductor is z, the depth-dependent field is written as

$$E_s(z) = 50e^{-\alpha z}e^{-j\beta z} = 50e^{-z/\delta}e^{-jz/\delta} = \underbrace{50\exp(-z/9.28)}_{\text{amplitude}}\exp(-j\underbrace{z/9.28}_{\text{angle}})$$

where z is in microns. Therefore, the plot of amplitude versus angle is simply a plot of e^{-x} versus x, where $x = z/9.28$; the starting amplitude is 50 and the $1/e$ amplitude (at $z = 9.28\ \mu$m) is 18.4.

11.25. A good conductor is planar in form and carries a uniform plane wave that has a wavelength of 0.3 mm and a velocity of 3×10^5 m/s. Assuming the conductor is non-magnetic, determine the frequency and the conductivity: First, we use

$$f = \frac{v}{\lambda} = \frac{3 \times 10^5}{3 \times 10^{-4}} = 10^9\ \text{Hz} = \underline{1\ \text{GHz}}$$

Next, for a good conductor,

$$\delta = \frac{\lambda}{2\pi} = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad \Rightarrow \quad \sigma = \frac{4\pi}{\lambda^2 f \mu} = \frac{4\pi}{(9 \times 10^{-8})(10^9)(4\pi \times 10^{-7})} = \underline{1.1 \times 10^5\ \text{S/m}}$$

11.26. The dimensions of a certain coaxial transmission line are $a = 0.8$mm and $b = 4$mm. The outer conductor thickness is 0.6mm, and all conductors have $\sigma = 1.6 \times 10^7$ S/m.

a) Find R, the resistance per unit length, at an operating frequency of 2.4 GHz: First

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(2.4 \times 10^8)(4\pi \times 10^{-7})(1.6 \times 10^7)}} = 2.57 \times 10^{-6}\text{m} = 2.57\mu\text{m}$$

Then, using (70) with a unit length, we find

$$R_{in} = \frac{1}{2\pi a \sigma \delta} = \frac{1}{2\pi(0.8 \times 10^{-3})(1.6 \times 10^7)(2.57 \times 10^{-6})} = 4.84\ \text{ohms/m}$$

The outer conductor resistance is then found from the inner through

$$R_{out} = \frac{a}{b}R_{in} = \frac{0.8}{4}(4.84) = 0.97\ \text{ohms/m}$$

The net resistance per length is then the sum, $R = R_{in} + R_{out} = \underline{5.81\ \text{ohms/m}}$.

11.26b. Use information from Secs. 5.10 and 9.10 to find C and L, the capacitance and inductance per unit length, respectively. The coax is air-filled. From those sections, we find (in free space)

$$C = \frac{2\pi\epsilon_0}{\ln(b/a)} = \frac{2\pi(8.854 \times 10^{-12})}{\ln(4/.8)} = \underline{3.46 \times 10^{-11} \text{ F/m}}$$

$$L = \frac{\mu_0}{2\pi}\ln(b/a) = \frac{4\pi \times 10^{-7}}{2\pi}\ln(4/.8) = \underline{3.22 \times 10^{-7} \text{ H/m}}$$

c) Find α and β if $\alpha + j\beta = \sqrt{j\omega C(R + j\omega L)}$: Taking real and imaginary parts of the given expression, we find

$$\alpha = \text{Re}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = \frac{\omega\sqrt{LC}}{\sqrt{2}}\left[\sqrt{1 + \left(\frac{R}{\omega L}\right)^2} - 1\right]^{1/2}$$

and

$$\beta = \text{Im}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = \frac{\omega\sqrt{LC}}{\sqrt{2}}\left[\sqrt{1 + \left(\frac{R}{\omega L}\right)^2} + 1\right]^{1/2}$$

These can be found by writing out $\alpha = \text{Re}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = (1/2)\sqrt{j\omega C(R + j\omega L)} + c.c.$, where $c.c$ denotes the complex conjugate. The result is squared, terms collected, and the square root taken. Now, using the values of R, C, and L found in parts a and b, we find $\alpha = \underline{3.0 \times 10^{-2} \text{ Np/m}}$ and $\beta = \underline{50.3 \text{ rad/m}}$.

11.27. The planar surface at $z = 0$ is a brass-Teflon interface. Use data available in Appendix C to evaluate the following ratios for a uniform plane wave having $\omega = 4 \times 10^{10}$ rad/s:

a) $\alpha_{\text{Tef}}/\alpha_{\text{brass}}$: From the appendix we find $\epsilon''/\epsilon' = .0003$ for Teflon, making the material a good dielectric. Also, for Teflon, $\epsilon'_R = 2.1$. For brass, we find $\sigma = 1.5 \times 10^7$ S/m, making brass a good conductor at the stated frequency. For a good dielectric (Teflon) we use the approximations:

$$\alpha \doteq \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon'}} = \left(\frac{\epsilon''}{\epsilon'}\right)\left(\frac{1}{2}\right)\omega\sqrt{\mu\epsilon'} = \frac{1}{2}\left(\frac{\epsilon''}{\epsilon'}\right)\frac{\omega}{c}\sqrt{\epsilon'_R}$$

$$\beta \doteq \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)\right] \doteq \omega\sqrt{\mu\epsilon'} = \frac{\omega}{c}\sqrt{\epsilon'_R}$$

For brass (good conductor) we have

$$\alpha \doteq \beta \doteq \sqrt{\pi f\mu\sigma_{\text{brass}}} = \sqrt{\pi\left(\frac{1}{2\pi}\right)(4 \times 10^{10})(4\pi \times 10^{-7})(1.5 \times 10^7)} = 6.14 \times 10^5 \text{ m}^{-1}$$

Now

$$\frac{\alpha_{\text{Tef}}}{\alpha_{\text{brass}}} = \frac{1/2\left(\epsilon''/\epsilon'\right)(\omega/c)\sqrt{\epsilon'_R}}{\sqrt{\pi f\mu\sigma_{\text{brass}}}} = \frac{(1/2)(.0003)(4 \times 10^{10}/3 \times 10^8)\sqrt{2.1}}{6.14 \times 10^5} = \underline{4.7 \times 10^{-8}}$$

b)

$$\frac{\lambda_{\text{Tef}}}{\lambda_{\text{brass}}} = \frac{(2\pi/\beta_{\text{Tef}})}{(2\pi/\beta_{\text{brass}})} = \frac{\beta_{\text{brass}}}{\beta_{\text{Tef}}} = \frac{c\sqrt{\pi f\mu\sigma_{\text{brass}}}}{\omega\sqrt{\epsilon'_{R\,\text{Tef}}}} = \frac{(3 \times 10^8)(6.14 \times 10^5)}{(4 \times 10^{10})\sqrt{2.1}} = \underline{3.2 \times 10^3}$$

11.27. (continued)

c)

$$\frac{v_{\text{Tef}}}{v_{\text{brass}}} = \frac{(\omega/\beta_{\text{Tef}})}{(\omega/\beta_{\text{brass}})} = \frac{\beta_{\text{brass}}}{\beta_{\text{Tef}}} = \underline{3.2 \times 10^3} \text{ as before}$$

11.28. A uniform plane wave in free space has electric field given by $\mathbf{E}_s = 10e^{-j\beta x}\mathbf{a}_z + 15e^{-j\beta x}\mathbf{a}_y$ V/m.

a) Describe the wave polarization: Since the two components have a fixed phase difference (in this case zero) with respect to time and position, the wave has linear polarization, with the field vector in the yz plane at angle $\phi = \tan^{-1}(10/15) = 33.7^\circ$ to the y axis.

b) Find $\mathbf{H}_s$: With propagation in forward x, we would have

$$\mathbf{H}_s = \frac{-10}{377}e^{-j\beta x}\mathbf{a}_y + \frac{15}{377}e^{-j\beta x}\mathbf{a}_z \text{ A/m} = \underline{-26.5e^{-j\beta x}\mathbf{a}_y + 39.8e^{-j\beta x}\mathbf{a}_z \text{ mA/m}}$$

c) determine the average power density in the wave in W/m^2: Use

$$\mathbf{P}_{avg} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\left[\frac{(10)^2}{377}\mathbf{a}_x + \frac{(15)^2}{377}\mathbf{a}_x\right] = 0.43\mathbf{a}_x \text{ W/m}^2 \text{ or } P_{avg} = \underline{0.43 \text{ W/m}^2}$$

11.29. Consider a left-circularly polarized wave in free space that propagates in the forward z direction. The electric field is given by the appropriate form of Eq. (80).

a) Determine the magnetic field phasor, $\mathbf{H}_s$:
We begin, using (80), with $\mathbf{E}_s = E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$. We find the two components of $\mathbf{H}_s$ separately, using the two components of $\mathbf{E}_s$. Specifically, the x component of $\mathbf{E}_s$ is associated with a y component of $\mathbf{H}_s$, and the y component of $\mathbf{E}_s$ is associated with a negative x component of $\mathbf{H}_s$. The result is

$$\mathbf{H}_s = \underline{\frac{E_0}{\eta_0}\left(\mathbf{a}_y - j\mathbf{a}_x\right)e^{-j\beta z}}$$

b) Determine an expression for the average power density in the wave in W/m^2 by direct application of Eq. (57): We have

$$\mathbf{P}_{z,avg} = \frac{1}{2}Re(\mathbf{E}_s \times \mathbf{H}_s^*) = \frac{1}{2}Re\left(E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z} \times \frac{E_0}{\eta_0}(\mathbf{a}_y - j\mathbf{a}_x)e^{+j\beta z}\right)$$
$$= \underline{\frac{E_0^2}{\eta_0}\mathbf{a}_z \text{ W/m}^2} \text{ (assuming } E_0 \text{ is real)}$$

11.30. The electric field of a uniform plane wave in free space is given by $\mathbf{E}_s = 10(\mathbf{a}_y + j\mathbf{a}_z)e^{-j25x}$.

a) Determine the frequency, f: Use

$$f = \frac{\beta c}{2\pi} = \frac{(25)(3\times 10^8)}{2\pi} = \underline{1.2 \text{ GHz}}$$

b) Find the magnetic field phasor, $\mathbf{H}_s$: With the Poynting vector in the positive x direction, a positive y component for $\mathbf{E}$ requires a positive z component for $\mathbf{H}$. Similarly, a positive z component for $\mathbf{E}$ requires a negative y component for $\mathbf{H}$. Therefore,

$$\underline{\mathbf{H}_s = \frac{10}{\eta_0}\left[\mathbf{a}_z - j\mathbf{a}_y\right]e^{-j25x}}$$

c) Describe the polarization of the wave: This is most clearly seen by first converting the given field to real instantaneous form:

$$\mathbf{E}(x,t) = \text{Re}\left\{\mathbf{E}_s e^{j\omega t}\right\} = 10\left[\cos(\omega t - 25x)\mathbf{a}_y - \sin(\omega t - 25x)\mathbf{a}_z\right]$$

At $x = 0$, this becomes,

$$\mathbf{E}(0,t) = 10\left[\cos(\omega t)\mathbf{a}_y - \sin(\omega t)\mathbf{a}_z\right]$$

With the wave traveling in the forward x direction, we recognize the polarization as <u>left circular</u>.

11.31. A linearly-polarized uniform plane wave, propagating in the forward z direction, is input to a lossless *anisotropic* material, in which the dielectric constant encountered by waves polarized along y (ϵ_{Ry}) differs from that seen by waves polarized along x (ϵ_{Rx}). Suppose $\epsilon_{Rx} = 2.15$, $\epsilon_{Ry} = 2.10$, and the wave electric field at input is polarized at 45° to the positive x and y axes. *Assume free space wavelength* λ.

a) Determine the shortest length of the material such that the wave as it emerges from the output end is circularly polarized: With the input field at 45°, the x and y components are of equal magnitude, and circular polarization will result if the phase difference between the components is $\pi/2$. Our requirement over length L is thus $\beta_x L - \beta_y L = \pi/2$, or

$$L = \frac{\pi}{2(\beta_x - \beta_y)} = \frac{\pi c}{2\omega(\sqrt{\epsilon_{Rx}} - \sqrt{\epsilon_{Ry}})}$$

With the given values, we find,

$$L = \frac{(58.3)\pi c}{2\omega} = 58.3\,\frac{\lambda}{4} = \underline{14.6\,\lambda}$$

b) Will the output wave be right- or left-circularly-polarized? With the dielectric constant greater for x-polarized waves, the x component will lag the y component in time at the output. The field can thus be written as $\mathbf{E} = E_0(\mathbf{a}_y - j\mathbf{a}_x)$, which is <u>left circular polarization</u>.

11.32. Suppose that the length of the medium of Problem 11.31 is made to be *twice* that as determined in the problem. Describe the polarization of the output wave in this case: With the length doubled, a phase shift of π radians develops between the two components. At the input, we can write the field as $\mathbf{E}_s(0) = E_0(\mathbf{a}_x + \mathbf{a}_y)$. After propagating through length L, we would have,

$$\mathbf{E}_s(L) = E_0[e^{-j\beta_x L}\mathbf{a}_x + e^{-j\beta_y L}\mathbf{a}_y] = E_0 e^{-j\beta_x L}[\mathbf{a}_x + e^{-j(\beta_y-\beta_x)L}\mathbf{a}_y]$$

where $(\beta_y - \beta_x)L = -\pi$ (since $\beta_x > \beta_y$), and so $\mathbf{E}_s(L) = E_0 e^{-j\beta_x L}[\mathbf{a}_x - \mathbf{a}_y]$. With the reversal of the y component, the wave polarization is rotated by 90°, but is still linear polarization.

11.33. Given a wave for which $\mathbf{E}_s = 15e^{-j\beta z}\mathbf{a}_x + 18e^{-j\beta z}e^{j\phi}\mathbf{a}_y$ V/m, propagating in a medium characterized by complex intrinsic impedance, η.

a) Find $\mathbf{H}_s$: With the wave propagating in the forward z direction, we find:

$$\mathbf{H}_s = \frac{1}{\eta}\left[-18e^{j\phi}\mathbf{a}_x + 15\mathbf{a}_y\right]e^{-j\beta z} \text{ A/m}$$

b) Determine the average power density in W/m^2: We find

$$P_{z,avg} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\text{Re}\left\{\frac{(15)^2}{\eta^*} + \frac{(18)^2}{\eta^*}\right\} = 275\,\text{Re}\left\{\frac{1}{\eta^*}\right\} \text{ W/m}^2$$

11.34. Given the general elliptically-polarized wave as per Eq. (73):

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{j\phi}\mathbf{a}_y]e^{-j\beta z}$$

a) Show, using methods similar to those of Example 11.7, that a linearly polarized wave results when superimposing the given field and a phase-shifted field of the form:

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{-j\phi}\mathbf{a}_y]e^{-j\beta z}e^{j\delta}$$

where δ is a constant: Adding the two fields gives

$$\mathbf{E}_{s,tot} = \left[E_{x0}\left(1 + e^{j\delta}\right)\mathbf{a}_x + E_{y0}\left(e^{j\phi} + e^{-j\phi}e^{j\delta}\right)\mathbf{a}_y\right]e^{-j\beta z}$$

$$= \left[E_{x0}e^{j\delta/2}\underbrace{\left(e^{-j\delta/2} + e^{j\delta/2}\right)}_{2\cos(\delta/2)}\mathbf{a}_x + E_{y0}e^{j\delta/2}\underbrace{\left(e^{-j\delta/2}e^{j\phi} + e^{-j\phi}e^{j\delta/2}\right)}_{2\cos(\phi-\delta/2)}\mathbf{a}_y\right]e^{-j\beta z}$$

This simplifies to $\mathbf{E}_{s,tot} = 2\left[E_{x0}\cos(\delta/2)\mathbf{a}_x + E_{y0}\cos(\phi - \delta/2)\mathbf{a}_y\right]e^{j\delta/2}e^{-j\beta z}$, which is linearly polarized.

b) Find δ in terms of ϕ such that the resultant wave is polarized along x: By inspecting the part *a* result, we achieve a zero y component when $2\phi - \delta = \pi$ (or odd multiples of π).