CHAPTER 14

14.1. A parallel-plate waveguide is known to have a cutoff wavelength for the $m = 1$ TE and TM modes of $\lambda_{c1} = 0.4$ cm. The guide is operated at wavelength $\lambda = 1$ mm. How many modes propagate? The cutoff wavelength for mode m is $\lambda_{cm} = 2nd/m$, where n is the refractive index of the guide interior. For the first mode, we are given

$$\lambda_{c1} = \frac{2nd}{1} = 0.4 \text{ cm} \;\; \Rightarrow \;\; d = \frac{0.4}{2n} = \frac{0.2}{n} \text{ cm}$$

Now, for mode m to propagate, we require

$$\lambda \leq \frac{2nd}{m} = \frac{0.4}{m} \;\; \Rightarrow \;\; m \leq \frac{0.4}{\lambda} = \frac{0.4}{0.1} = 4$$

So, accounting for 2 modes (TE and TM) for each value of m, and the single TEM mode, we will have a total of $\underline{9}$ modes.

14.2. A parallel-plate guide is to be constructed for operation in the TEM mode only over the frequency range $0 < f < 3$ GHz. The dielectric between plates is to be teflon ($\epsilon'_R = 2.1$). Determine the maximum allowable plate separation, d: We require that $f < f_{c1}$, which, using (7), becomes

$$f < \frac{c}{2nd} \;\; \Rightarrow \;\; d_{max} = \frac{c}{2nf_{max}} = \frac{3 \times 10^8}{2\sqrt{2.1}\,(3 \times 10^9)} = \underline{3.45\,\text{cm}}$$

14.3. A lossless parallel-plate waveguide is known to propagate the $m = 2$ TE and TM modes at frequencies as low as 10GHz. If the plate separation is 1 cm, determine the dielectric constant of the medium between plates: Use

$$f_{c2} = \frac{c}{nd} = \frac{3 \times 10^{10}}{n(1)} = 10^{10} \;\; \Rightarrow \;\; n = 3 \;\text{ or }\; \epsilon_R = \underline{9}$$

14.4. A $d = 1$ cm parallel-plate guide is made with glass ($n = 1.45$) between plates. If the operating frequency is 32 GHz, which modes will propagate? For a propagating mode, we require $f > f_{cm}$ Using (7) and the given values, we write

$$f > \frac{mc}{2nd} \;\; \Rightarrow \;\; m < \frac{2fnd}{c} = \frac{2(32 \times 10^9)(1.45)(.01)}{3 \times 10^8} = 3.09$$

The maximum allowed m in this case is thus 3, and the propagating modes will be $\underline{TM_1}$, $\underline{TE_1}$, $\underline{TM_2}$, $\underline{TE_2}$, $\underline{TM_3}$, and $\underline{TE_3}$.

14.5. For the guide of Problem 14.4, and at the 32 GHz frequency, determine the difference between the group delays of the highest order mode (TE or TM) and the TEM mode. Assume a propagation distance of 10 cm: From Problem 14.4, we found $m_{max} = 3$. The group velocity of a TE or TM mode for $m = 3$ is

$$v_{g3} = \frac{c}{n}\sqrt{1 - \left(\frac{f_{c3}}{f}\right)^2} \quad \text{where} \quad f_{c3} = \frac{3(3 \times 10^{10})}{2(1.45)(1)} = 3.1 \times 10^{10} = 31\,\text{GHz}$$

14.5. (continued) Thus

$$v_{g3} = \frac{3 \times 10^{10}}{1.45}\sqrt{1 - \left(\frac{31}{32}\right)^2} = 5.13 \times 10^9 \text{ cm/s}$$

For the TEM mode (assuming no material dispersion) $v_{g,TEM} = c/n = 3 \times 10^{10}/1.45 = 2.07 \times 10^{10}$ cm/s. The group delay difference is now

$$\Delta t_g = z\left(\frac{1}{v_{g3}} - \frac{1}{v_{g,TEM}}\right) = 10\left(\frac{1}{5.13 \times 10^9} - \frac{1}{2.07 \times 10^{10}}\right) = \underline{1.5 \text{ ns}}$$

14.6. The cutoff frequency of the $m = 1$ TE and TM modes in a parallel-plate guide is known to be $f_{c1} = 7.5$ GHz. The guide is used at wavelength $\lambda = 1.5$ cm. Find the group velocity of the $m = 2$ TE and TM modes. First we know that $f_{c2} = 2f_{c1} = 15$ GHz. Then $f = c/\lambda = 3 \times 10^8/.015 = 20$ GHz. Now, using (23),

$$v_{g2} = \frac{c}{n}\sqrt{1 - \left(\frac{f_{c2}}{f}\right)^2} = \frac{c}{n}\sqrt{1 - \left(\frac{15}{20}\right)^2} = \underline{2 \times 10^8/n \text{ m/s}}$$

n was not specified in the problem.

14.7. A parallel-plate guide is partially filled with two lossless dielectrics (Fig. 14.23) where $\epsilon'_{R1} = 4.0$, $\epsilon'_{R2} = 2.1$, and $d = 1$ cm. At a certain frequency, it is found that the TM$_1$ mode propagates through the guide without suffering any reflective loss at the dielectric interface.

a) Find this frequency: The ray angle is such that the wave is incident on the interface at Brewster's angle. In this case

$$\theta_B = \tan^{-1}\sqrt{\frac{2.1}{4.0}} = 35.9^\circ$$

The ray angle is thus $\theta = 90 - 35.9 = 54.1^\circ$. The cutoff frequency for the $m = 1$ mode is

$$f_{c1} = \frac{c}{2d\sqrt{\epsilon'_{R1}}} = \frac{3 \times 10^{10}}{2(1)(2)} = 7.5\,\text{GHz}$$

The frequency is thus $f = f_{c1}/\cos\theta = 7.5/\cos(54.1^\circ) = \underline{12.8\,\text{GHz}}$.

b) Is the guide operating at a single TM mode at the frequency found in part *a*? The cutoff frequency for the next higher mode, TM$_2$ is $f_{c2} = 2f_{c1} = 15$ GHz. The 12.8 GHz operating frequency is below this, so TM$_2$ will not propagate. So the answer is yes.

14.8. In the guide of Problem 14.7, it is found that $m = 1$ modes propagating from left to right totally reflect at the interface, so that no power is transmitted into the region of dielectric constant ϵ'_{R2}.

a) Determine the range of frequencies over which this will occur: For total reflection, the ray angle measured from the normal to the interface must be greater than or equal to the critical angle, θ_c, where $\sin\theta_c = (\epsilon'_{R2}/\epsilon'_{R1})^{1/2}$. The *minimum* mode ray angle is then $\theta_{1\,min} = 90^\circ - \theta_c$. Now, using (5), we write

$$90^\circ - \theta_c = \cos^{-1}\left(\frac{\pi}{k_{min}d}\right) = \cos^{-1}\left(\frac{\pi c}{2\pi f_{min} d\sqrt{4}}\right) = \cos^{-1}\left(\frac{c}{4df_{min}}\right)$$

14.8a. (continued)

Now

$$\cos(90-\theta_c) = \sin\theta_c = \sqrt{\frac{\epsilon'_{R2}}{\epsilon'_{R1}}} = \frac{c}{4df_{min}}$$

Therefore $f_{min} = c/(2\sqrt{2.1}d) = (3\times 10^8)/(2\sqrt{2.1}(.01)) = 10.35\,\text{GHz}$. The frequency range is thus $\underline{f > 10.35\,\text{GHz}}$.

b) Does your part a answer in any way relate to the cutoff frequency for $m = 1$ modes in any region? We note that $f_{min} = c/(2\sqrt{2.1}d) = f_{c1}$ in guide 2. To summarize, as frequency is lowered, the ray angle in guide 1 decreases, which leads to the incident angle at the interface increasing to eventually reach and surpass the critical angle. At the critical angle, the refracted angle in guide 2 is 90°, which corresponds to a zero degree ray angle in that guide. This defines the cutoff condition in guide 2. So it would make sense that $f_{min} = f_{c1}(\text{guide 2})$.

14.9. A rectangular waveguide has dimensions $a = 6$ cm and $b = 4$ cm.

a) Over what range of frequencies will the guide operate single mode? The cutoff frequency for mode mp is, using Eq. (54):

$$f_{c,mn} = \frac{c}{2n}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{p}{b}\right)^2}$$

where n is the refractive index of the guide interior. We require that the frequency lie between the cutoff frequencies of the TE_{10} and TE_{01} modes. These will be:

$$f_{c10} = \frac{c}{2na} = \frac{3\times 10^8}{2n(.06)} = \frac{2.5\times 10^9}{n}$$

$$f_{c01} = \frac{c}{2nb} = \frac{3\times 10^8}{2n(.04)} = \frac{3.75\times 10^9}{n}$$

Thus, the range of frequencies over which single mode operation will occur is

$$\underline{\frac{2.5}{n}\,\text{GHz} < f < \frac{3.75}{n}\,\text{GHz}}$$

b) Over what frequency range will the guide support *both* TE_{10} and TE_{01} modes and no others? We note first that f must be greater than f_{c01} to support both modes, but must be less than the cutoff frequency for the next higher order mode. This will be f_{c11}, given by

$$f_{c11} = \frac{c}{2n}\sqrt{\left(\frac{1}{.06}\right)^2 + \left(\frac{1}{.04}\right)^2} = \frac{30c}{2n} = \frac{4.5\times 10^9}{n}$$

The allowed frequency range is then

$$\underline{\frac{3.75}{n}\,\text{GHz} < f < \frac{4.5}{n}\,\text{GHz}}$$

14.10. Two rectangular waveguides are joined end-to-end. The guides have identical dimensions, where $a = 2b$. One guide is air-filled; the other is filled with a lossless dielectric characterized by ϵ'_R.

a) Determine the maximum allowable value of ϵ'_R such that single mode operation can be simultaneously ensured in *both* guides at some frequency: Since $a = 2b$, the cutoff frequency for any mode in either guide is written using (54):

$$f_{cmp} = \sqrt{\left(\frac{mc}{4nb}\right)^2 + \left(\frac{pc}{2nb}\right)^2}$$

where $n = 1$ in guide 1 and $n = \sqrt{\epsilon'_R}$ in guide 2. We see that, with $a = 2b$, the next modes (having the next higher cutoff frequency) above TE_{10} with be TE_{20} and TE_{01}. We also see that in general, $f_{cmp}(\text{guide 2}) < f_{cmp}(\text{guide 1})$. To assure single mode operation in both guides, the operating frequency must be above cutoff for TE_{10} in both guides, and below cutoff for the next mode in both guides. The allowed frequency range is therefore $f_{c10}(\text{guide 1}) < f < f_{c20}(\text{guide 2})$. This leads to $c/(2a) < f < c/(a\sqrt{\epsilon'_R})$. For this range to be viable, it is required that $\underline{\epsilon'_R < 4}$.

b) Write an expression for the frequency range over which single mode operation will occur in both guides; your answer should be in terms of ϵ'_R, guide dimensions as needed, and other known constants: This was already found in part *a*:

$$\underline{\frac{c}{2a} < f < \frac{c}{\sqrt{\epsilon'_R}\, a}}$$

where $\epsilon'_R < 4$.

14.11. An air-filled rectangular waveguide is to be constructed for single-mode operation at 15 GHz. Specify the guide dimensions, a and b, such that the design frequency is 10/while being 10% lower than the cutoff frequency for the next higher-order mode: For an air-filled guide, we have

$$f_{c,mp} = \sqrt{\left(\frac{mc}{2a}\right)^2 + \left(\frac{pc}{2b}\right)^2}$$

For TE_{10} we have $f_{c10} = c/2a$, while for the next mode (TE_{01}), $f_{c01} = c/2b$. Our requirements state that $f = 1.1 f_{c10} = 0.9 f_{c01}$. So $f_{c10} = 15/1.1 = 13.6$ GHz and $f_{c01} = 15/0.9 = 16.7$ GHz. The guide dimensions will be

$$a = \frac{c}{2f_{c10}} = \frac{3 \times 10^{10}}{2(13.6 \times 10^9)} = \underline{1.1\text{ cm}} \quad \text{and} \quad b = \frac{c}{2f_{c01}} = \frac{3 \times 10^{10}}{2(16.7 \times 10^9)} = \underline{0.90\text{ cm}}$$

14.12. Using the relation $P_{av} = (1/2)\text{Re}\{\mathbf{E}_s \times \mathbf{H}_s^*\}$, and Eqs. (44) through (46), show that the average power density in the TE_{10} mode in a rectangular waveguide is given by

$$P_{av} = \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\, \mathbf{a}_z \qquad \text{W/m}^2$$

(note that the sin term is erroneously to the first power in the original problem statement). Inspecting (44) through (46), we see that (46) includes a factor of j, and so would lead to an imaginary part of the total power when the cross product with E_y is taken. Therefore, the real power in this case is found through the cross product of (44) with the complex conjugate of (45), or

$$P_{av} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_{ys} \times \mathbf{H}^*_{xs}\right\} = \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\,\mathbf{a}_z \qquad \text{W/m}^2$$

14.13. Integrate the result of Problem 14.12 over the guide cross-section $0 < x < a$, $0 < y < b$, to show that the power in Watts transmitted down the guide is given as

$$P = \frac{\beta_{10} ab}{4\omega\mu} E_0^2 = \frac{ab}{4\eta} E_0^2 \sin\theta_{10} \ \text{ W}$$

where $\eta = \sqrt{\mu/\epsilon}$ (note misprint in problem statement), and θ_{10} is the wave angle associated with the TE_{10} mode. Interpret. First, the integration:

$$P = \int_0^b \int_0^a \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\,\mathbf{a}_z \cdot \mathbf{a}_z\, dx\, dy = \frac{\beta_{10} ab}{4\omega\mu} E_0^2$$

Next, from (20), we have $\beta_{10} = \omega\sqrt{\mu\epsilon}\sin\theta_{10}$, which, on substitution, leads to

$$P = \frac{ab}{4\eta} E_0^2 \sin\theta_{10} \ \text{ W} \ \text{ with } \eta = \sqrt{\frac{\mu}{\epsilon}}$$

The $\sin\theta_{10}$ dependence demonstrates the principle of group velocity as energy velocity (or power). This was considered in the discussion leading to Eq. (23).

14.14. Show that the group dispersion parameter, $d^2\beta/d\omega^2$, for given mode in a parallel-plate or rectangular waveguide is given by

$$\frac{d^2\beta}{d\omega^2} = -\frac{n}{\omega c}\left(\frac{\omega_c}{\omega}\right)^2 \left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}$$

where ω_c is the radian cutoff frequency for the mode in question (note that the first derivative form was already found, resulting in Eq. (23)). First, taking the reciprocal of (23), we find

$$\frac{d\beta}{d\omega} = \frac{n}{c}\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-1/2}$$

Taking the derivative of this equation with respect to ω leads to

$$\frac{d^2\beta}{d\omega^2} = \frac{n}{c}\left(-\frac{1}{2}\right)\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}\left(\frac{2\omega_c^2}{\omega^3}\right) = -\frac{n}{\omega c}\left(\frac{\omega_c}{\omega}\right)^2\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}$$

14.15. Consider a transform-limited pulse of center frequency $f = 10$ GHz and of full-width $2T = 1.0$ ns. The pulse propagates in a lossless single mode rectangular guide which is air-filled and in which the 10 GHz operating frequency is 1.1 times the cutoff frequency of the TE_{10} mode. Using the result of Problem 14.14, determine the length of the guide over which the pulse broadens to twice its initial width: The broadened pulse will have width given by $T' = \sqrt{T^2 + (\Delta\tau)^2}$, where $\Delta\tau = \beta_2 L/T$ for a transform limited pulse (assumed gaussian). β_2 is the Problem 14.14 result evaluated at the operating frequency, or

$$\beta_2 = \frac{d^2\beta}{d\omega^2}\Big|_{\omega=10\,\text{GHz}} = -\frac{1}{(2\pi \times 10^{10})(3 \times 10^8)}\left(\frac{1}{1.1}\right)^2\left[1 - \left(\frac{1}{1.1}\right)^2\right]^{-3/2}$$
$$= 6.1 \times 10^{-19}\ \text{s}^2/\text{m} = 0.61\ \text{ns}^2/\text{m}$$

Now $\Delta\tau = 0.61L/0.5 = 1.2L$ ns. For the pulse width to double, we have $T' = 1$ ns, and

$$\sqrt{(.05)^2 + (1.2L)^2} = 1 \quad \Rightarrow \quad L = 0.72\,\text{m} = \underline{72\,\text{cm}}$$

14.15. (continued)

What simple step can be taken to reduce the amount of pulse broadening in this guide, while maintaining the same initial pulse width? It can be seen that β_2 can be reduced by increasing the operating frequency relative to the cutoff frequency; i.e., operate as far above cutoff as possible, without allowing the next higher-order modes to propagate.

14.16. A symmetric dielectric slab waveguide has a slab thickness $d = 10\ \mu$m, with $n_1 = 1.48$ and $n_2 = 1.45$. If the operating wavelength is $\lambda = 1.3\ \mu$m, what modes will propagate? We use the condition expressed through (77): $k_0 d\sqrt{n_1^2 - n_2^2} \geq (m-1)\pi$. Since $k_0 = 2\pi/\lambda$, the condition becomes

$$\frac{2d}{\lambda}\sqrt{n_1^2 - n_2^2} \geq (m-1) \quad \Rightarrow \quad \frac{2(10)}{1.3}\sqrt{(1.48)^2 - (1.45)^2} = 4.56 \geq m-1$$

Therefore, $m_{max} = 5$, and we have TE and TM modes for which $m = 1, 2, 3, 4, 5$ propagating (ten total).

14.17. A symmetric slab waveguide is known to support only a single pair of TE and TM modes at wavelength $\lambda = 1.55\ \mu$m. If the slab thickness is $5\ \mu$m, what is the maximum value of n_1 if $n_2 = 3.3$ (assume 3.30)? Using (78) we have

$$\frac{2\pi d}{\lambda}\sqrt{n_1^2 - n_2^2} < \pi \quad \Rightarrow \quad n_1 < \sqrt{\frac{\lambda}{2d} + n_2^2} = \sqrt{\frac{1.55}{2(5)} + (3.30)^2} = \underline{3.32}$$

14.18. $n_1 = 1.50$, $n_2 = 1.45$, and $d = 10\ \mu$m in a symmetric slab waveguide (note that the index values were reversed in the original problem statement).

a) What is the phase velocity of the $m = 1$ TE or TM mode at cutoff? At cutoff, the mode propagates in the slab at the critical angle, which means that the phase velocity will be equal to that of a plane wave in the upper or lower media of index n_2. Phase velocity will therefore be $v_p(\text{cutoff}) = c/n_2 = 3 \times 10^8/1.45 = \underline{2.07 \times 10^8\ \text{m/s}}$.

b) What is the phase velocity of the $m = 2$ TE or TM modes at cutoff? The reasoning of part *a* applies to all modes, so the answer is the same, or $\underline{2.07 \times 10^8\ \text{m/s}}$.

14.19. An *asymmetric* slab waveguide is shown in Fig. 14.24. In this case, the regions above and below the slab have unequal refractive indices, where $n_1 > n_3 > n_2$ (note error in problem statement).

a) Write, in terms of the appropriate indices, an expression for the minimum possible wave angle, θ_1, that a guided mode may have: The wave angle must be equal to or greater than the critical angle of total reflection at *both* interfaces. The minimum wave angle is thus determined by the *greater* of the two critical angles. Since $n_3 > n_2$, we find $\theta_{min} = \theta_{c,13} = \underline{\sin^{-1}(n_3/n_1)}$.

b) Write an expression for the maximum phase velocity a guided mode may have in this structure, using given or known parameters: We have $v_{p,max} = \omega/\beta_{min}$, where $\beta_{min} = n_1 k_0 \sin\theta_{1,min} = n_1 k_0 n_3/n_1 = n_3 k_0$. Thus $v_{p,max} = \omega/(n_3 k_0) = \underline{c/n_3}$.

14.20. A step index optical fiber is known to be single mode at wavelengths $\lambda > 1.2\ \mu$m. Another fiber is to be fabricated from the same materials, but is to be single mode at wavelengths $\lambda > 0.63\ \mu$m. By what percentage must the core radius of the new fiber differ from the old one, and should it be larger or smaller? We use the cutoff condition, given by (80):

$$\lambda > \frac{2\pi a}{2.405}\sqrt{n_1^2 - n_2^2}$$

14.20. (continued) With λ reduced, the core radius, a, must also be reduced by the same fraction. Therefore, the percentage *reduction* required in the core radius will be

$$\% = \frac{1.2 - .63}{1.2} \times 100 = \underline{47.5\%}$$

14.21. A short dipole carrying current $I_0 \cos\omega t$ in the $\mathbf{a}_z$ direction is located at the origin in free space.

a) If $\beta = 1$ rad/m, $r = 2$ m, $\theta = 45^\circ$, $\phi = 0$, and $t = 0$, give a unit vector in rectangular components that shows the instantaneous direction of $\mathbf{E}$: In spherical coordinates, the components of $\mathbf{E}$ are given by (82) and (83):

$$E_r = \frac{I_0 d\eta}{2\pi} \cos\theta e^{-j2\pi r/\lambda} \left(\frac{1}{r^2} + \frac{\lambda}{j2\pi r^3} \right)$$

$$E_\theta = \frac{I_0 d\eta}{4\pi} \sin\theta e^{-j2\pi r/\lambda} \left(j\frac{2\pi}{\lambda r} + \frac{1}{r^2} + \frac{\lambda}{j2\pi r^3} \right)$$

Since we want a unit vector at $t = 0$, we need only the relative amplitudes of the two components, but we need the absolute phases. Since $\theta = 45^\circ$, $\sin\theta = \cos\theta = 1/\sqrt{2}$. Also, with $\beta = 1 = 2\pi/\lambda$, it follows that $\lambda = 2\pi$ m. The above two equations can be simplified by these substitutions, while dropping all amplitude terms that are common to both. Obtain

$$A_r = \left(\frac{1}{r^2} + \frac{1}{jr^3} \right) e^{-jr}$$

$$A_\theta = \frac{1}{2} \left(j\frac{1}{r} + \frac{1}{r^2} + \frac{1}{jr^3} \right) e^{-jr}$$

Now with $r = 2$ m, we obtain

$$A_r = \left(\frac{1}{4} - j\frac{1}{8} \right) e^{-j2} = \frac{1}{4}(1.12)e^{-j26.6^\circ} e^{-j2}$$

$$A_\theta = \left(j\frac{1}{4} + \frac{1}{8} - j\frac{1}{16} \right) e^{-j2} = \frac{1}{4}(0.90)e^{j56.3^\circ} e^{-j2}$$

The total vector is now $\mathbf{A} = A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta$. We can normalize the vector by first finding the magnitude:

$$|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}^*} = \frac{1}{4}\sqrt{(1.12)^2 + (0.90)^2} = 0.359$$

Dividing the field vector by this magnitude and converting 2 rad to 114.6°, we write the normalized vector as

$$\mathbf{A}_{Ns} = 0.780e^{-j141.2^\circ}\mathbf{a}_r + 0.627e^{-58.3^\circ}\mathbf{a}_\theta$$

In real instantaneous form, this becomes

$$\mathbf{A}_N(t) = \mathrm{Re}\left(\mathbf{A}_{Ns}e^{j\omega t} \right) = 0.780\cos(\omega t - 141.2^\circ)\mathbf{a}_r + 0.627\cos(\omega t - 58.3^\circ)\mathbf{a}_\theta$$

We evaluate this at $t = 0$ to find

$$\mathbf{A}_N(0) = 0.780\cos(141.2^\circ)\mathbf{a}_r + 0.627\cos(58.3^\circ)\mathbf{a}_\theta = -0.608\mathbf{a}_r + 0.330\mathbf{a}_\theta$$

14.21a. (continued)

Dividing by the magnitude, $\sqrt{(0.608)^2 + (0.330)^2} = 0.692$, we obtain the unit vector at $t = 0$: $\mathbf{a}_N(0) = -0.879\mathbf{a}_r + 0.477\mathbf{a}_\theta$. We next convert this to cartesian components:

$$a_{Nx} = \mathbf{a}_N(0) \cdot \mathbf{a}_x = -0.879 \sin\theta \cos\phi + 0.477 \cos\theta \cos\phi = \frac{1}{\sqrt{2}}(-0.879 + 0.477) = -0.284$$

$$a_{Ny} = \mathbf{a}_N(0) \cdot \mathbf{a}_y = -0.879 \sin\theta \sin\phi + 0.477 \cos\theta \sin\phi = 0 \;\text{ since } \phi = 0$$

$$a_{Nz} = \mathbf{a}_N(0) \cdot \mathbf{a}_z = -0.879 \cos\theta - 0.477 \sin\theta = \frac{1}{\sqrt{2}}(-0.879 - 0.477) = -0.959$$

The final result is then

$$\mathbf{a}_N(0) = \underline{-0.284\mathbf{a}_x - 0.959\mathbf{a}_z}$$

b) What fraction of the total average power is radiated in the belt, $80^\circ < \theta < 100^\circ$? We use the far-zone phasor fields, (84) and (85), and first find the average power density:

$$P_{avg} = \frac{1}{2}\mathrm{Re}[E_{\theta s} H_{\phi s}^*] = \frac{I_0^2 d^2 \eta}{8\lambda^2 r^2} \sin^2\theta \;\; \mathrm{W/m^2}$$

We integrate this over the given belt, an at radius r:

$$P_{belt} = \int_0^{2\pi} \int_{80^\circ}^{100^\circ} \frac{I_0^2 d^2 \eta}{8\lambda^2 r^2} \sin^2\theta \, r^2 \sin\theta \, d\theta \, d\phi = \frac{\pi I_0^2 d^2 \eta}{4\lambda^2} \int_{80^\circ}^{100^\circ} \sin^3\theta \, d\theta$$

Evaluating the integral, we find

$$P_{belt} = \frac{\pi I_0^2 d^2 \eta}{4\lambda^2} \left[-\frac{1}{3} \cos\theta \left(\sin^2\theta + 2 \right) \right]_{80}^{100} = (0.344) \frac{\pi I_0^2 d^2 \eta}{4\lambda^2}$$

The total power is found by performing the same integral over θ, where $0 < \theta < 180^\circ$. Doing this, it is found that

$$P_{tot} = (1.333) \frac{\pi I_0^2 d^2 \eta}{4\lambda^2}$$

The fraction of the total power in the belt is then $f = 0.344/1.333 = \underline{0.258}$.

14.22. Prepare a curve, r vs. θ in polar coordinates, showing the locus in the $\phi = 0$ plane where:

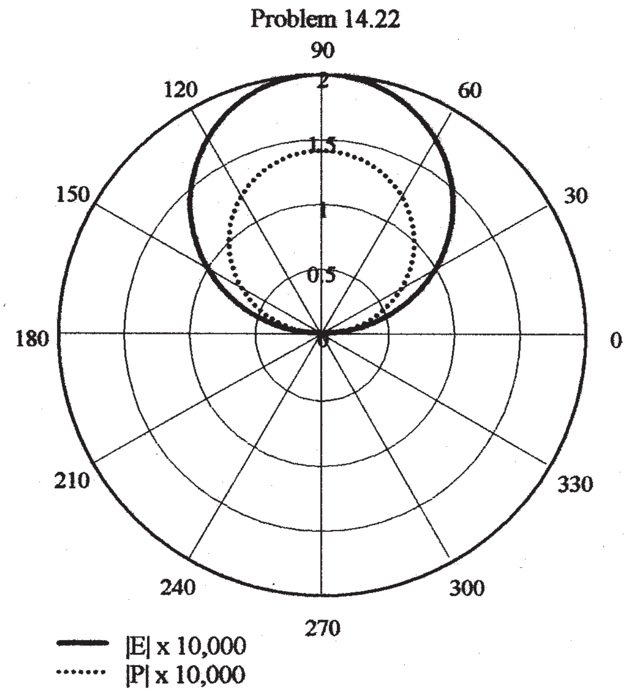
a) the radiation field $|E_{\theta s}|$ is one-half of its value at $r = 10^4$ m, $\theta = \pi/2$: Assuming the far field approximation, we use (84) to set up the equation:

$$|E_{\theta s}| = \frac{I_0 d\eta}{2\lambda r} \sin\theta = \frac{1}{2} \times \frac{I_0 d\eta}{2 \times 10^4 \lambda} \;\; \Rightarrow \;\; r = 2 \times 10^4 \sin\theta$$

b) the average radiated power density, $P_{r,av}$, is one-half of its value at $r = 10^4$ m, $\theta = \pi/2$. To find the average power, we use (84) and (85) in

$$P_{r,av} = \frac{1}{2}\mathrm{Re}\{E_{\theta s} H_{\phi s}^*\} = \frac{1}{2} \frac{I_0^2 d^2 \eta}{4\lambda^2 r^2} \sin^2\theta = \frac{1}{2} \times \frac{1}{2} \frac{I_0^2 d^2 \eta}{4\lambda^2 (10^8)} \;\; \Rightarrow \;\; r = \sqrt{2} \times 10^4 \sin\theta$$

14.22. (continued) The polar plots for field ($r = 2 \times 10^4 \sin\theta$) and power ($r = \sqrt{2} \times 10^4 \sin\theta$) are shown below. Both are circles.



14.23. Two short antennas at the origin in free space carry identical currents of $5\cos\omega t$ A, one in the $\mathbf{a}_z$ direction, one in the $\mathbf{a}_y$ direction. Let $\lambda = 2\pi$ m and $d = 0.1$ m. Find $\mathbf{E}_s$ at the distant point:

a) ($x = 0, y = 1000, z = 0$): This point lies along the axial direction of the $\mathbf{a}_y$ antenna, so its contribution to the field will be zero. This leaves the $\mathbf{a}_z$ antenna, and since $\theta = 90°$, only the $E_{\theta s}$ component will be present (as (82) and (83) show). Since we are in the far zone, (84) applies. We use $\theta = 90°$, $d = 0.1$, $\lambda = 2\pi$, $\eta = \eta_0 = 120\pi$, and $r = 1000$ to write:

$$\mathbf{E}_s = E_{\theta s}\mathbf{a}_\theta = j\frac{I_0 d\eta}{2\lambda r}\sin\theta e^{-j2\pi r/\lambda}\,\mathbf{a}_\theta = j\frac{5(0.1)(120\pi)}{4\pi(1000)}e^{-j1000}\,\mathbf{a}_\theta$$
$$= j(1.5 \times 10^{-2})e^{-j1000}\,\mathbf{a}_\theta = \underline{-j(1.5 \times 10^{-2})e^{-j1000}\,\mathbf{a}_z \text{ V/m}}$$

b) (0, 0, 1000): Along the z axis, only the $\mathbf{a}_y$ antenna will contribute to the field. Since the distance is the same, we can apply the part a result, modified such the the field direction is in $-\mathbf{a}_y$: $\mathbf{E}_s = \underline{-j(1.5 \times 10^{-2})e^{-j1000}\,\mathbf{a}_y \text{ V/m}}$

c) (1000, 0, 0): Here, both antennas will contribute. Applying the results of parts a and b, we find $\mathbf{E}_s = \underline{-j(1.5 \times 10^{-2})(\mathbf{a}_y + \mathbf{a}_z)}$.

d) Find $\mathbf{E}$ at (1000, 0, 0) at $t = 0$: This is found through

$$\mathbf{E}(t) = \text{Re}\left(\mathbf{E}_s e^{j\omega t}\right) = (1.5 \times 10^{-2})\sin(\omega t - 1000)(\mathbf{a}_y + \mathbf{a}_z)$$

Evaluating at $t = 0$, we find
$\mathbf{E}(0) = (1.5 \times 10^{-2})[-\sin(1000)](\mathbf{a}_y + \mathbf{a}_z) = \underline{-(1.24 \times 10^{-2})(\mathbf{a}_y + \mathbf{a}_z) \text{ V/m}}$.

e) Find $|\mathbf{E}|$ at (1000, 0, 0) at $t = 0$: Taking the magnitude of the part d result, we find $|\mathbf{E}| = \underline{1.75 \times 10^{-2} \text{ V/m}}$.

14.24. A short current element has $d = 0.03\lambda$. Calculate the radiation resistance for each of the following current distributions:

a) uniform: In this case, (86) applies directly and we find

$$R_{rad} = 80\pi^2 \left(\frac{d}{\lambda}\right)^2 = 80\pi^2 (.03)^2 = \underline{0.711\ \Omega}$$

b) linear, $I(z) = I_0(0.5d - |z|)/0.5d$: Here, the average current is $0.5I_0$, and so the average power drops by a factor of 0.25. The radiation resistance therefore is down to one-fourth the value found in part a, or $R_{rad} = (0.25)(0.711) = \underline{0.178\ \Omega}$.

c) step, I_0 for $0 < |z| < 0.25d$ and $0.5I_0$ for $0.25d < |z| < 0.5d$: In this case the average current on the wire is $0.75I_0$. The radiated power (and radiation resistance) are down to a factor of $(0.75)^2$ times their values for a uniform current, and so $R_{rad} = (0.75)^2(0.711) = \underline{0.400\ \Omega}$.

14.25. A dipole antenna in free space has a linear current distribution. If the length is 0.02λ, what value of I_0 is required to:

a) provide a radiation-field amplitude of 100 mV/m at a distance of one mile, at $\theta = 90^\circ$: With a linear current distribution, the peak current, I_0, occurs at the center of the dipole; current decreases linearly to zero at the two ends. The average current is thus $I_0/2$, and we use Eq. (84) to write:

$$|E_\theta| = \frac{I_0 d \eta_0}{4\lambda r} \sin(90^\circ) = \frac{I_0(0.02)(120\pi)}{(4)(5280)(12)(0.0254)} = 0.1 \quad \Rightarrow \quad I_0 = \underline{85.4\ \text{A}}$$

b) radiate a total power of 1 watt? We use

$$P_{avg} = \left(\frac{1}{4}\right)\left(\frac{1}{2} I_0^2 R_{rad}\right)$$

where the radiation resistance is given by Eq. (86), and where the factor of 1/4 arises from the average current of $I_0/2$: We obtain $P_{avg} = 10\pi^2 I_0^2 (0.02)^2 = 1 \quad \Rightarrow \quad I_0 = \underline{5.03\ \text{A}}$.

14.26. A monopole antenna in free space, extending vertically over a perfectly conducting plane, has a linear current distribution. If the length of the antenna is 0.01λ, what value of I_0 is required to

a) provide a radiation field amplitude of 100 mV/m at a distance of 1 mi, at $\theta = 90^\circ$: The image antenna below the plane provides a radiation pattern that is identical to a dipole antenna of length 0.02λ. The radiation field is thus given by (84) in free space, where $\theta = 90^\circ$, and with an additional factor of $1/2$ included to account for the linear current distribution:

$$|E_\theta| = \frac{1}{2}\frac{I_0 d \eta_0}{2\lambda r} \quad \Rightarrow \quad I_0 = \frac{4r|E_\theta|}{(d/\lambda)\eta_0} = \frac{4(5289)(12 \times .0254)(100 \times 10^{-3})}{(.02)(377)} = \underline{85.4\ \text{A}}$$

b) radiate a total power of 1W: For the monopole over the conducting plane, power is radiated only over the upper half-space. This reduces the radiation resistance of the equivalent dipole antenna by a factor of one-half. Additionally, the linear current distribution reduces the radiation resistance of a dipole having uniform current by a factor of one-fourth. Therefore, R_{rad} is one-eighth the value obtained from (86), or $R_{rad} = 10\pi^2(d/\lambda)^2$. The current magnitude is now

$$I_0 = \left[\frac{2P_{av}}{R_{rad}}\right]^{1/2} = \left[\frac{2(1)}{10\pi^2(d/\lambda)^2}\right]^{1/2} = \frac{\sqrt{2}}{\sqrt{10}\,\pi(.02)} = \underline{7.1\ \text{A}}$$

14.27. The radiation field of a certain short vertical current element is $E_{\theta s} = (20/r)\sin\theta\, e^{-j10\pi r}$ V/m if it is located at the origin in free space.

a) Find $E_{\theta s}$ at $P(r = 100, \theta = 90^\circ, \phi = 30^\circ)$: Substituting these values into the given formula, find

$$E_{\theta s} = \frac{20}{100}\sin(90^\circ)e^{-j10\pi(100)} = \underline{0.2e^{-j1000\pi}\ \mathrm{V/m}}$$

b) Find $E_{\theta s}$ at P if the vertical element is located at $A(0.1, 90^\circ, 90^\circ)$: This places the element on the y axis at $y = 0.1$. As a result of moving the antenna from the origin to $y = 0.1$, the change in distance to point P is negligible when considering the change in field *amplitude*, but is not when considering the change in *phase*. Consider lines drawn from the origin to P and from $y = 0.1$ to P. These lines can be considered essentially parallel, and so the difference in their lengths is $l \doteq 0.1\sin(30^\circ)$, with the line from $y = 0.1$ being shorter by this amount. The construction and arguments are similar to those used in the discussion of the electric dipole in Sec. 4.7. The electric field is now the result of part a, modified by including a shorter distance, r, in the phase term only. We show this as an additional phase factor:

$$E_{\theta s} = 0.2e^{-j1000\pi}e^{j10\pi(0.1\sin 30} = \underline{0.2e^{-j1000\pi}e^{j0.5\pi}\ \mathrm{V/m}}$$

c) Find $E_{\theta s}$ at P if identical elements are located at $A(0.1, 90^\circ, 90^\circ)$ and $B(0.1, 90^\circ, 270^\circ)$: The original element of part b is still in place, but a new one has been added at $y = -0.1$. Again, constructing a line between B and P, we find, using the same arguments as in part b, that the length of this line is approximately $0.1\sin(30^\circ)$ *longer* than the distance from the origin to P. The part b result is thus modified to include the contribution from the second element, whose field will add to that of the first:

$$E_{\theta s} = 0.2e^{-j1000\pi}\left(e^{j0.5\pi} + e^{-j0.5\pi}\right) = 0.2e^{-j1000\pi}\,2\cos(0.5\pi) = \underline{0}$$

The two fields are out of phase at P under the approximations we have used.