CHAPTER 2

2.1. Four 10nC positive charges are located in the $z = 0$ plane at the corners of a square 8cm on a side. A fifth 10nC positive charge is located at a point 8cm distant from the other charges. Calculate the magnitude of the total force on this fifth charge for $\epsilon = \epsilon_0$:

Arrange the charges in the xy plane at locations (4,4), (4,-4), (-4,4), and (-4,-4). Then the fifth charge will be on the z axis at location $z = 4\sqrt{2}$, which puts it at 8cm distance from the other four. By symmetry, the force on the fifth charge will be z-directed, and will be four times the z component of force produced by each of the four other charges.

$$F = \frac{4}{\sqrt{2}} \times \frac{q^2}{4\pi\epsilon_0 d^2} = \frac{4}{\sqrt{2}} \times \frac{(10^{-8})^2}{4\pi(8.85 \times 10^{-12})(0.08)^2} = \underline{4.0 \times 10^{-4}\ \text{N}}$$

2.2. A charge $Q_1 = 0.1\ \mu$C is located at the origin, while $Q_2 = 0.2\ \mu$C is at $A(0.8, -0.6, 0)$. Find the locus of points in the $z = 0$ plane at which the x component of the force on a third positive charge is zero.

To solve this problem, the z coordinate of the third charge is immaterial, so we can place it in the xy plane at coordinates $(x, y, 0)$. We take its magnitude to be Q_3. The vector directed from the first charge to the third is $\mathbf{R}_{13} = x\mathbf{a}_x + y\mathbf{a}_y$; the vector directed from the second charge to the third is $\mathbf{R}_{23} = (x - 0.8)\mathbf{a}_x + (y + 0.6)\mathbf{a}_y$. The force on the third charge is now

$$\mathbf{F}_3 = \frac{Q_3}{4\pi\epsilon_0}\left[\frac{Q_1\mathbf{R}_{13}}{|\mathbf{R}_{13}|^3} + \frac{Q_2\mathbf{R}_{23}}{|\mathbf{R}_{23}|^3}\right]$$
$$= \frac{Q_3 \times 10^{-6}}{4\pi\epsilon_0}\left[\frac{0.1(x\mathbf{a}_x + y\mathbf{a}_y)}{(x^2 + y^2)^{1.5}} + \frac{0.2[(x - 0.8)\mathbf{a}_x + (y + 0.6)\mathbf{a}_y]}{[(x - 0.8)^2 + (y + 0.6)^2]^{1.5}}\right]$$

We desire the x component to be zero. Thus,

$$0 = \left[\frac{0.1x\mathbf{a}_x}{(x^2 + y^2)^{1.5}} + \frac{0.2(x - 0.8)\mathbf{a}_x}{[(x - 0.8)^2 + (y + 0.6)^2]^{1.5}}\right]$$

or

$$\underline{x[(x - 0.8)^2 + (y + 0.6)^2]^{1.5} = 2(0.8 - x)(x^2 + y^2)^{1.5}}$$

2.3. Point charges of 50nC each are located at $A(1, 0, 0)$, $B(-1, 0, 0)$, $C(0, 1, 0)$, and $D(0, -1, 0)$ in free space. Find the total force on the charge at A.

The force will be:

$$\mathbf{F} = \frac{(50 \times 10^{-9})^2}{4\pi\epsilon_0}\left[\frac{\mathbf{R}_{CA}}{|\mathbf{R}_{CA}|^3} + \frac{\mathbf{R}_{DA}}{|\mathbf{R}_{DA}|^3} + \frac{\mathbf{R}_{BA}}{|\mathbf{R}_{BA}|^3}\right]$$

where $\mathbf{R}_{CA} = \mathbf{a}_x - \mathbf{a}_y$, $\mathbf{R}_{DA} = \mathbf{a}_x + \mathbf{a}_y$, and $\mathbf{R}_{BA} = 2\mathbf{a}_x$. The magnitudes are $|\mathbf{R}_{CA}| = |\mathbf{R}_{DA}| = \sqrt{2}$, and $|\mathbf{R}_{BA}| = 2$. Substituting these leads to

$$\mathbf{F} = \frac{(50 \times 10^{-9})^2}{4\pi\epsilon_0}\left[\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{2}{8}\right]\mathbf{a}_x = \underline{21.5\mathbf{a}_x\ \mu\text{N}}$$

where distances are in meters.

2.4. Let $Q_1 = 8\ \mu$C be located at $P_1(2, 5, 8)$ while $Q_2 = -5\ \mu$C is at $P_2(6, 15, 8)$. Let $\epsilon = \epsilon_0$.

a) Find $\mathbf{F}_2$, the force on Q_2: This force will be

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0} \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|^3} = \frac{(8 \times 10^{-6})(-5 \times 10^{-6})}{4\pi\epsilon_0} \frac{(4\mathbf{a}_x + 10\mathbf{a}_y)}{(116)^{1.5}} = \underline{(-1.15\mathbf{a}_x - 2.88\mathbf{a}_y)\,\text{mN}}$$

b) Find the coordinates of P_3 if a charge Q_3 experiences a total force $\mathbf{F}_3 = 0$ at P_3: This force in general will be:

$$\mathbf{F}_3 = \frac{Q_3}{4\pi\epsilon_0}\left[\frac{Q_1\mathbf{R}_{13}}{|\mathbf{R}_{13}|^3} + \frac{Q_2\mathbf{R}_{23}}{|\mathbf{R}_{23}|^3}\right]$$

where $\mathbf{R}_{13} = (x-2)\mathbf{a}_x + (y-5)\mathbf{a}_y$ and $\mathbf{R}_{23} = (x-6)\mathbf{a}_x + (y-15)\mathbf{a}_y$. Note, however, that all three charges must lie in a straight line, and the location of Q_3 will be along the vector $\mathbf{R}_{12}$ extended past Q_2. The slope of this vector is $(15-5)/(6-2) = 2.5$. Therefore, we look for P_3 at coordinates $(x, 2.5x, 8)$. With this restriction, the force becomes:

$$\mathbf{F}_3 = \frac{Q_3}{4\pi\epsilon_0}\left[\frac{8[(x-2)\mathbf{a}_x + 2.5(x-2)\mathbf{a}_y]}{[(x-2)^2 + (2.5)^2(x-2)^2]^{1.5}} - \frac{5[(x-6)\mathbf{a}_x + 2.5(x-6)\mathbf{a}_y]}{[(x-6)^2 + (2.5)^2(x-6)^2]^{1.5}}\right]$$

where we require the term in large brackets to be zero. This leads to

$$8(x-2)[((2.5)^2+1)(x-6)^2]^{1.5} - 5(x-6)[((2.5)^2+1)(x-2)^2]^{1.5} = 0$$

which reduces to

$$8(x-6)^2 - 5(x-2)^2 = 0$$

or

$$x = \frac{6\sqrt{8} - 2\sqrt{5}}{\sqrt{8} - \sqrt{5}} = \underline{21.1}$$

The coordinates of P_3 are thus $\underline{P_3(21.1, 52.8, 8)}$

2.5. Let a point charge $Q_1 25$ nC be located at $P_1(4, -2, 7)$ and a charge $Q_2 = 60$ nC be at $P_2(-3, 4, -2)$.

a) If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at $P_3(1, 2, 3)$: This field will be

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25\mathbf{R}_{13}}{|\mathbf{R}_{13}|^3} + \frac{60\mathbf{R}_{23}}{|\mathbf{R}_{23}|^3}\right]$$

where $\mathbf{R}_{13} = -3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{R}_{23} = 4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{41}$ and $|\mathbf{R}_{23}| = \sqrt{45}$. So

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25 \times (-3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z)}{(41)^{1.5}} + \frac{60 \times (4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z)}{(45)^{1.5}}\right]$$
$$= \underline{4.58\mathbf{a}_x - 0.15\mathbf{a}_y + 5.51\mathbf{a}_z}$$

b) At what point on the y axis is $E_x = 0$? P_3 is now at $(0, y, 0)$, so $\mathbf{R}_{13} = -4\mathbf{a}_x + (y+2)\mathbf{a}_y - 7\mathbf{a}_z$ and $\mathbf{R}_{23} = 3\mathbf{a}_x + (y-4)\mathbf{a}_y + 2\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{65 + (y+2)^2}$ and $|\mathbf{R}_{23}| = \sqrt{13 + (y-4)^2}$. Now the x component of $\mathbf{E}$ at the new P_3 will be:

$$E_x = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25 \times (-4)}{[65 + (y+2)^2]^{1.5}} + \frac{60 \times 3}{[13 + (y-4)^2]^{1.5}}\right]$$

To obtain $E_x = 0$, we require the expression in the large brackets to be zero. This expression simplifies to the following quadratic:

$$0.48y^2 + 13.92y + 73.10 = 0$$

which yields the two values: $y = \underline{-6.89, -22.11}$

2.6. Point charges of 120 nC are located at $A(0, 0, 1)$ and $B(0, 0, -1)$ in free space.

a) Find **E** at $P(0.5, 0, 0)$: This will be

$$\mathbf{E}_P = \frac{120 \times 10^{-9}}{4\pi\epsilon_0}\left[\frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3} + \frac{\mathbf{R}_{BP}}{|\mathbf{R}_{BP}|^3}\right]$$

where $\mathbf{R}_{AP} = 0.5\mathbf{a}_x - \mathbf{a}_z$ and $\mathbf{R}_{BP} = 0.5\mathbf{a}_x + \mathbf{a}_z$. Also, $|\mathbf{R}_{AP}| = |\mathbf{R}_{BP}| = \sqrt{1.25}$. Thus:

$$\mathbf{E}_P = \frac{120 \times 10^{-9}\mathbf{a}_x}{4\pi(1.25)^{1.5}\epsilon_0} = \underline{772 \text{ V/m}}$$

b) What single charge at the origin would provide the identical field strength? We require

$$\frac{Q_0}{4\pi\epsilon_0(0.5)^2} = 772$$

from which we find $Q_0 = \underline{21.5 \text{ nC}}$.

2.7. A 2 μC point charge is located at $A(4, 3, 5)$ in free space. Find E_ρ, E_ϕ, and E_z at $P(8, 12, 2)$. Have

$$\mathbf{E}_P = \frac{2 \times 10^{-6}}{4\pi\epsilon_0}\frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3} = \frac{2 \times 10^{-6}}{4\pi\epsilon_0}\left[\frac{4\mathbf{a}_x + 9\mathbf{a}_y - 3\mathbf{a}_z}{(106)^{1.5}}\right] = 65.9\mathbf{a}_x + 148.3\mathbf{a}_y - 49.4\mathbf{a}_z$$

Then, at point P, $\rho = \sqrt{8^2 + 12^2} = 14.4$, $\phi = \tan^{-1}(12/8) = 56.3^\circ$, and $z = z$. Now,

$$E_\rho = \mathbf{E}_p \cdot \mathbf{a}_\rho = 65.9(\mathbf{a}_x \cdot \mathbf{a}_\rho) + 148.3(\mathbf{a}_y \cdot \mathbf{a}_\rho) = 65.9\cos(56.3^\circ) + 148.3\sin(56.3^\circ) = \underline{159.7}$$

and

$$E_\phi = \mathbf{E}_p \cdot \mathbf{a}_\phi = 65.9(\mathbf{a}_x \cdot \mathbf{a}_\phi) + 148.3(\mathbf{a}_y \cdot \mathbf{a}_\phi) = -65.9\sin(56.3^\circ) + 148.3\cos(56.3^\circ) = \underline{27.4}$$

Finally, $E_z = \underline{-49.4}$

2.8. Given point charges of -1 μC at $P_1(0, 0, 0.5)$ and $P_2(0, 0, -0.5)$, and a charge of 2 μC at the origin, find **E** at $P(0, 2, 1)$ in spherical components, assuming $\epsilon = \epsilon_0$.

The field will take the general form:

$$\mathbf{E}_P = \frac{10^{-6}}{4\pi\epsilon_0}\left[-\frac{\mathbf{R}_1}{|\mathbf{R}_1|^3} + \frac{2\mathbf{R}_2}{|\mathbf{R}_2|^3} - \frac{\mathbf{R}_3}{|\mathbf{R}_3|^3}\right]$$

where $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$ are the vectors to P from each of the charges in their original listed order. Specifically, $\mathbf{R}_1 = (0, 2, 0.5)$, $\mathbf{R}_2 = (0, 2, 1)$, and $\mathbf{R}_3 = (0, 2, 1.5)$. The magnitudes are $|\mathbf{R}_1| = 2.06$, $|\mathbf{R}_2| = 2.24$, and $|\mathbf{R}_3| = 2.50$. Thus

$$\mathbf{E}_P = \frac{10^{-6}}{4\pi\epsilon_0}\left[\frac{-(0, 2, 0.5)}{(2.06)^3} + \frac{2(0, 2, 1)}{(2.24)^3} - \frac{(0, 2, 1.5)}{(2.50)^3}\right] = 89.9\mathbf{a}_y + 179.8\mathbf{a}_z$$

Now, at P, $r = \sqrt{5}$, $\theta = \cos^{-1}(1/\sqrt{5}) = 63.4^\circ$, and $\phi = 90^\circ$. So

$$E_r = \mathbf{E}_P \cdot \mathbf{a}_r = 89.9(\mathbf{a}_y \cdot \mathbf{a}_r) + 179.8(\mathbf{a}_z \cdot \mathbf{a}_r) = 89.9\sin\theta\sin\phi + 179.8\cos\theta = \underline{160.9}$$

$$E_\theta = \mathbf{E}_P \cdot \mathbf{a}_\theta = 89.9(\mathbf{a}_y \cdot \mathbf{a}_\theta) + 179.8(\mathbf{a}_z \cdot \mathbf{a}_\theta) = 89.9\cos\theta\sin\phi + 179.8(-\sin\theta) = \underline{-120.5}$$

$$E_\phi = \mathbf{E}_P \cdot \mathbf{a}_\phi = 89.9(\mathbf{a}_y \cdot \mathbf{a}_\phi) + 179.8(\mathbf{a}_z \cdot \mathbf{a}_\phi) = 89.9\cos\phi = \underline{0}$$

2.9. A 100 nC point charge is located at $A(-1, 1, 3)$ in free space.

a) Find the locus of all points $P(x, y, z)$ at which $E_x = 500$ V/m: The total field at P will be:

$$\mathbf{E}_P = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3}$$

where $\mathbf{R}_{AP} = (x+1)\mathbf{a}_x + (y-1)\mathbf{a}_y + (z-3)\mathbf{a}_z$, and where $|\mathbf{R}_{AP}| = [(x+1)^2 + (y-1)^2 + (z-3)^2]^{1/2}$. The x component of the field will be

$$E_x = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \left[\frac{(x+1)}{[(x+1)^2 + (y-1)^2 + (z-3)^2]^{1.5}} \right] = 500 \text{ V/m}$$

And so our condition becomes:

$$\underline{(x+1) = 0.56\,[(x+1)^2 + (y-1)^2 + (z-3)^2]^{1.5}}$$

b) Find y_1 if $P(-2, y_1, 3)$ lies on that locus: At point P, the condition of part a becomes

$$3.19 = \left[1 + (y_1 - 1)^2\right]^3$$

from which $(y_1 - 1)^2 = 0.47$, or $y_1 = \underline{1.69 \text{ or } 0.31}$

2.10. Charges of 20 and -20 nC are located at $(3, 0, 0)$ and $(-3, 0, 0)$, respectively. Let $\epsilon = \epsilon_0$. Determine $|\mathbf{E}|$ at $P(0, y, 0)$: The field will be

$$\mathbf{E}_P = \frac{20 \times 10^{-9}}{4\pi\epsilon_0} \left[\frac{\mathbf{R}_1}{|\mathbf{R}_1|^3} - \frac{\mathbf{R}_2}{|\mathbf{R}_2|^3} \right]$$

where $\mathbf{R}_1$, the vector from the positive charge to point P is $(-3, y, 0)$, and $\mathbf{R}_2$, the vector from the negative charge to point P, is $(3, y, 0)$. The magnitudes of these vectors are $|\mathbf{R}_1| = |\mathbf{R}_2| = \sqrt{9 + y^2}$. Substituting these into the expression for $\mathbf{E}_P$ produces

$$\mathbf{E}_P = \frac{20 \times 10^{-9}}{4\pi\epsilon_0} \left[\frac{-6\mathbf{a}_x}{(9 + y^2)^{1.5}} \right]$$

from which

$$|\mathbf{E}_P| = \underline{\frac{1079}{(9 + y^2)^{1.5}} \text{ V/m}}$$

2.11. A charge Q_0 located at the origin in free space produces a field for which $E_z = 1$ kV/m at point $P(-2, 1, -1)$.

a) Find Q_0: The field at P will be

$$\mathbf{E}_P = \frac{Q_0}{4\pi\epsilon_0} \left[\frac{-2\mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z}{6^{1.5}} \right]$$

Since the z component is of value 1 kV/m, we find $Q_0 = -4\pi\epsilon_0 6^{1.5} \times 10^3 = \underline{-1.63\ \mu\text{C}}$.

2.11. (continued)

b) Find **E** at $M(1, 6, 5)$ in cartesian coordinates: This field will be:

$$\mathbf{E}_M = \frac{-1.63 \times 10^{-6}}{4\pi\epsilon_0} \left[\frac{\mathbf{a}_x + 6\mathbf{a}_y + 5\mathbf{a}_z}{[1 + 36 + 25]^{1.5}} \right]$$

or $\mathbf{E}_M = \underline{-30.11\mathbf{a}_x - 180.63\mathbf{a}_y - 150.53\mathbf{a}_z}$.

c) Find **E** at $M(1, 6, 5)$ in cylindrical coordinates: At M, $\rho = \sqrt{1 + 36} = 6.08$, $\phi = \tan^{-1}(6/1) = 80.54^\circ$, and $z = 5$. Now

$$E_\rho = \mathbf{E}_M \cdot \mathbf{a}_\rho = -30.11 \cos\phi - 180.63 \sin\phi = -183.12$$

$$E_\phi = \mathbf{E}_M \cdot \mathbf{a}_\phi = -30.11(-\sin\phi) - 180.63 \cos\phi = 0 \quad \text{(as expected)}$$

so that $\mathbf{E}_M = \underline{-183.12\mathbf{a}_\rho - 150.53\mathbf{a}_z}$.

d) Find **E** at $M(1, 6, 5)$ in spherical coordinates: At M, $r = \sqrt{1 + 36 + 25} = 7.87$, $\phi = 80.54^\circ$ (as before), and $\theta = \cos^{-1}(5/7.87) = 50.58^\circ$. Now, since the charge is at the origin, we expect to obtain only a radial component of $\mathbf{E}_M$. This will be:

$$E_r = \mathbf{E}_M \cdot \mathbf{a}_r = -30.11 \sin\theta \cos\phi - 180.63 \sin\theta \sin\phi - 150.53 \cos\theta = \underline{-237.1}$$

2.12. The volume charge density $\rho_v = \rho_0 e^{-|x|-|y|-|z|}$ exists over all free space. Calculate the total charge present: This will be 8 times the integral of ρ_v over the first octant, or

$$Q = 8 \int_0^\infty \int_0^\infty \int_0^\infty \rho_0 e^{-x-y-z} \, dx \, dy \, dz = \underline{8\rho_0}$$

2.13. A uniform volume charge density of 0.2 $\mu\text{C/m}^3$ (note typo in book) is present throughout the spherical shell extending from $r = 3$ cm to $r = 5$ cm. If $\rho_v = 0$ elsewhere:

a) find the total charge present throughout the shell: This will be

$$Q = \int_0^{2\pi} \int_0^\pi \int_{.03}^{.05} 0.2\, r^2 \sin\theta \, dr \, d\theta \, d\phi = \left[4\pi(0.2)\frac{r^3}{3} \right]_{.03}^{.05} = 8.21 \times 10^{-5} \, \mu\text{C} = \underline{82.1 \text{ pC}}$$

b) find r_1 if half the total charge is located in the region $3\,\text{cm} < r < r_1$: If the integral over r in part a is taken to r_1, we would obtain

$$\left[4\pi(0.2)\frac{r^3}{3} \right]_{.03}^{r_1} = 4.105 \times 10^{-5}$$

Thus

$$r_1 = \left[\frac{3 \times 4.105 \times 10^{-5}}{0.2 \times 4\pi} + (.03)^3 \right]^{1/3} = \underline{4.24 \text{ cm}}$$

2.14. Let

$$\rho_v = 5e^{-0.1\rho}\ (\pi - |\phi|)\ \frac{1}{z^2+10}\ \ \mu\text{C/m}^3$$

in the region $0 \le \rho \le 10$, $-\pi < \phi < \pi$, all z, and $\rho_v = 0$ elsewhere.

a) Determine the total charge present: This will be the integral of ρ_v over the region where it exists; specifically,

$$Q = \int_{-\infty}^{\infty}\int_{-\pi}^{\pi}\int_{0}^{10}\ 5e^{-0.1\rho}\ (\pi - |\phi|)\ \frac{1}{z^2+10}\rho\, d\rho\, d\phi\, dz$$

which becomes

$$Q = 5\left[\frac{e^{-0.1\rho}}{(0.1)^2}(-0.1-1)\right]_0^{10}\int_{-\infty}^{\infty} 2\int_0^{\pi}\ (\pi - \phi)\ \frac{1}{z^2+10}d\phi\, dz$$

or

$$Q = 5 \times 26.4 \int_{-\infty}^{\infty}\pi^2\frac{1}{z^2+10}\,dz$$

Finally,

$$Q = 5 \times 26.4 \times \pi^2\left[\frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{z}{\sqrt{10}}\right)\right]_{-\infty}^{\infty} = \frac{5(26.4)\pi^3}{\sqrt{10}} = 1.29 \times 10^3\ \mu\text{C} = \underline{1.29\text{ mC}}$$

b) Calculate the charge within the region $0 \le \rho \le 4$, $-\pi/2 < \phi < \pi/2$, $-10 < z < 10$: With the limits thus changed, the integral for the charge becomes:

$$Q' = \int_{-10}^{10} 2\int_0^{\pi/2}\int_0^{4}\ 5e^{-0.1\rho}\ (\pi - \phi)\ \frac{1}{z^2+10}\rho\, d\rho\, d\phi\, dz$$

Following the same evaulation procedure as in part a, we obtain $Q' = \underline{0.182\text{ mC}}$.

2.15. A spherical volume having a 2 μm radius contains a uniform volume charge density of 10^{15} C/m^3.

a) What total charge is enclosed in the spherical volume?
This will be $Q = (4/3)\pi(2 \times 10^{-6})^3 \times 10^{15} = \underline{3.35 \times 10^{-2}\text{ C}}$.

b) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3mm on a side, and that there is no charge between spheres. What is the average volume charge density throughout this large region? Each cube will contain the equivalent of one little sphere. Neglecting the little sphere volume, the average density becomes

$$\rho_{v,avg} = \frac{3.35 \times 10^{-2}}{(0.003)^3} = \underline{1.24 \times 10^6\text{ C/m}^3}$$

2.16. The region in which $4 < r < 5$, $0 < \theta < 25^\circ$, and $0.9\pi < \phi < 1.1\pi$ contains the volume charge density of $\rho_v = 10(r-4)(r-5)\sin\theta\sin(\phi/2)$. Outside the region, $\rho_v = 0$. Find the charge within the region: The integral that gives the charge will be

$$Q = 10\int_{.9\pi}^{1.1\pi}\int_0^{25^\circ}\int_4^5 (r-4)(r-5)\sin\theta\sin(\phi/2)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

2.16. (continued) Carrying out the integral, we obtain

$$Q = 10\left[\frac{r^5}{5} - 9\frac{r^4}{4} + 20\frac{r^3}{3}\right]_4^5 \left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{25^\circ} \left[-2\cos\left(\frac{\theta}{2}\right)\right]_{.9\pi}^{1.1\pi}$$
$$= 10(-3.39)(.0266)(.626) = \underline{0.57}\ \text{C}$$

2.17. A uniform line charge of 16 nC/m is located along the line defined by $y = -2$, $z = 5$. If $\epsilon = \epsilon_0$:

a) Find **E** at $P(1, 2, 3)$: This will be

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\frac{\mathbf{R}_P}{|\mathbf{R}_P|^2}$$

where $\mathbf{R}_P = (1, 2, 3) - (1, -2, 5) = (0, 4, -2)$, and $|\mathbf{R}_P|^2 = 20$. So

$$\mathbf{E}_P = \frac{16 \times 10^{-9}}{2\pi\epsilon_0}\left[\frac{4\mathbf{a}_y - 2\mathbf{a}_z}{20}\right] = \underline{57.5\mathbf{a}_y - 28.8\mathbf{a}_z\ \text{V/m}}$$

b) Find **E** at that point in the $z = 0$ plane where the direction of **E** is given by $(1/3)\mathbf{a}_y - (2/3)\mathbf{a}_z$: With $z = 0$, the general field will be

$$\mathbf{E}_{z=0} = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(y+2)\mathbf{a}_y - 5\mathbf{a}_z}{(y+2)^2 + 25}\right]$$

We require $|E_z| = -|2E_y|$, so $2(y + 2) = 5$. Thus $y = 1/2$, and the field becomes:

$$\mathbf{E}_{z=0} = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{2.5\mathbf{a}_y - 5\mathbf{a}_z}{(2.5)^2 + 25}\right] = \underline{23\mathbf{a}_y - 46\mathbf{a}_z}$$

2.18. Uniform line charges of 0.4 μC/m and -0.4 μC/m are located in the $x = 0$ plane at $y = -0.6$ and $y = 0.6$ m respectively. Let $\epsilon = \epsilon_0$.

a) Find **E** at $P(x, 0, z)$: In general, we have

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{\mathbf{R}_{+P}}{|\mathbf{R}_{+P}|} - \frac{\mathbf{R}_{-P}}{|\mathbf{R}_{-P}|}\right]$$

where $\mathbf{R}_{+P}$ and $\mathbf{R}_{-P}$ are, respectively, the vectors directed from the positive and negative line charges to the point P, and these are normal to the z axis. We thus have $\mathbf{R}_{+P} = (x, 0, z) - (0, -.6, z) = (x, .6, 0)$, and $\mathbf{R}_{-P} = (x, 0, z) - (0, .6, z) = (x, -.6, 0)$. So

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{x\mathbf{a}_x + 0.6\mathbf{a}_y}{x^2 + (0.6)^2} - \frac{x\mathbf{a}_x - 0.6\mathbf{a}_y}{x^2 + (0.6)^2}\right] = \frac{0.4 \times 10^{-6}}{2\pi\epsilon_0}\left[\frac{1.2\mathbf{a}_y}{x^2 + 0.36}\right] = \underline{\frac{8.63\mathbf{a}_y}{x^2 + 0.36}\ \text{kV/m}}$$

2.18. (continued)

b) Find $\mathbf{E}$ at $Q(2, 3, 4)$: This field will in general be:

$$\mathbf{E}_Q = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{\mathbf{R}_{+Q}}{|\mathbf{R}_{+Q}|} - \frac{\mathbf{R}_{-Q}}{|\mathbf{R}_{-Q}|}\right]$$

where $\mathbf{R}_{+Q} = (2, 3, 4) - (0, -.6, 4) = (2, 3.6, 0)$, and $\mathbf{R}_{-Q} = (2, 3, 4) - (0, .6, 4) = (2, 2.4, 0)$. Thus

$$\mathbf{E}_Q = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{2\mathbf{a}_x + 3.6\mathbf{a}_y}{2^2 + (3.6)^2} - \frac{2\mathbf{a}_x + 2.4\mathbf{a}_y}{2^2 + (2.4)^2}\right] = \underline{-625.8\mathbf{a}_x - 241.6\mathbf{a}_y} \text{ V/m}$$

2.19. A uniform line charge of 2 μC/m is located on the z axis. Find $\mathbf{E}$ in cartesian coordinates at $P(1, 2, 3)$ if the charge extends from

a) $-\infty < z < \infty$: With the infinite line, we know that the field will have only a radial component in cylindrical coordinates (or x and y components in cartesian). The field from an infinite line on the z axis is generally $\mathbf{E} = [\rho_l/(2\pi\epsilon_0\rho)]\mathbf{a}_\rho$. Therefore, at point P:

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\frac{\mathbf{R}_{zP}}{|\mathbf{R}_{zP}|^2} = \frac{(2\times 10^{-6})}{2\pi\epsilon_0}\;\frac{\mathbf{a}_x + 2\mathbf{a}_y}{5} = \underline{7.2\mathbf{a}_x + 14.4\mathbf{a}_y \text{ kV/m}}$$

where $\mathbf{R}_{zP}$ is the vector that extends from the line charge to point P, and is perpendicular to the z axis; i.e., $\mathbf{R}_{zP} = (1, 2, 3) - (0, 0, 3) = (1, 2, 0)$.

b) $-4 \le z \le 4$: Here we use the general relation

$$\mathbf{E}_P = \int \frac{\rho_l dz}{4\pi\epsilon_0}\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{r}' = z\mathbf{a}_z$. So the integral becomes

$$\mathbf{E}_P = \frac{(2\times 10^{-6})}{4\pi\epsilon_0}\int_{-4}^{4}\frac{\mathbf{a}_x + 2\mathbf{a}_y + (3 - z)\mathbf{a}_z}{[5 + (3 - z)^2]^{1.5}}\,dz$$

Using integral tables, we obtain:

$$\mathbf{E}_P = 3597\left[\frac{(\mathbf{a}_x + 2\mathbf{a}_y)(z - 3) + 5\mathbf{a}_z}{(z^2 - 6z + 14)}\right]_{-4}^{4} \text{ V/m} = \underline{4.9\mathbf{a}_x + 9.8\mathbf{a}_y + 4.9\mathbf{a}_z \text{ kV/m}}$$

The student is invited to verify that when evaluating the above expression over the limits $-\infty < z < \infty$, the z component vanishes and the x and y components become those found in part a.

2.20. Uniform line charges of 120 nC/m lie along the entire extent of the three coordinate axes. Assuming free space conditions, find $\mathbf{E}$ at $P(-3, 2, -1)$: Since all line charges are infinitely-long, we can write:

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{\mathbf{R}_{xP}}{|\mathbf{R}_{xP}|^2} + \frac{\mathbf{R}_{yP}}{|\mathbf{R}_{yP}|^2} + \frac{\mathbf{R}_{zP}}{|\mathbf{R}_{zP}|^2}\right]$$

where $\mathbf{R}_{xP}$, $\mathbf{R}_{yP}$, and $\mathbf{R}_{zP}$ are the normal vectors from each of the three axes that terminate on point P. Specifically, $\mathbf{R}_{xP} = (-3, 2, -1) - (-3, 0, 0) = (0, 2, -1)$, $\mathbf{R}_{yP} = (-3, 2, -1) - (0, 2, 0) = (-3, 0, -1)$, and $\mathbf{R}_{zP} = (-3, 2, -1) - (0, 0, -1) = (-3, 2, 0)$. Substituting these into the expression for $\mathbf{E}_P$ gives

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{2\mathbf{a}_y - \mathbf{a}_z}{5} + \frac{-3\mathbf{a}_x - \mathbf{a}_z}{10} + \frac{-3\mathbf{a}_x + 2\mathbf{a}_y}{13}\right] = \underline{-1.15\mathbf{a}_x + 1.20\mathbf{a}_y - 0.65\mathbf{a}_z \text{ kV/m}}$$

2.21. Two identical uniform line charges with $\rho_l = 75$ nC/m are located in free space at $x = 0$, $y = \pm 0.4$ m. What force per unit length does each line charge exert on the other? The charges are parallel to the z axis and are separated by 0.8 m. Thus the field from the charge at $y = -0.4$ evaluated at the location of the charge at $y = +0.4$ will be $\mathbf{E} = [\rho_l/(2\pi\epsilon_0(0.8))]\mathbf{a}_y$. The force on a differential length of the line at the positive y location is $d\mathbf{F} = dq\mathbf{E} = \rho_l dz\mathbf{E}$. Thus the force per unit length acting on the line at postive y arising from the charge at negative y is

$$\mathbf{F} = \int_0^1 \frac{\rho_l^2\, dz}{2\pi\epsilon_0(0.8)}\,\mathbf{a}_y = 1.26 \times 10^{-4}\,\mathbf{a}_y \text{ N/m} = \underline{126\,\mathbf{a}_y\ \mu\text{N/m}}$$

The force on the line at negative y is of course the same, but with $-\mathbf{a}_y$.

2.22. A uniform surface charge density of $5\,\text{nC/m}^2$ is present in the region $x = 0$, $-2 < y < 2$, and all z. If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at:

a) $P_A(3, 0, 0)$: We use the superposition integral:

$$\mathbf{E} = \int\int \frac{\rho_s da}{4\pi\epsilon_0}\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} = 3\mathbf{a}_x$ and $\mathbf{r}' = y\mathbf{a}_y + z\mathbf{a}_z$. The integral becomes:

$$\mathbf{E}_{PA} = \frac{\rho_s}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\int_{-2}^{2}\frac{3\mathbf{a}_x - y\mathbf{a}_y - z\mathbf{a}_z}{[9 + y^2 + z^2]^{1.5}}\,dy\,dz$$

Since the integration limits are symmetric about the origin, and since the y and z components of the integrand exhibit odd parity (change sign when crossing the origin, but otherwise symmetric), these will integrate to zero, leaving only the x component. This is evident just from the symmetry of the problem. Performing the z integration first on the x component, we obtain (using tables):

$$E_{x,PA} = \frac{3\rho_s}{4\pi\epsilon_0}\int_{-2}^{2}\frac{dy}{(9 + y^2)}\left[\frac{z}{\sqrt{9 + y^2 + z^2}}\right]_{-\infty}^{\infty} = \frac{3\rho_s}{2\pi\epsilon_0}\int_{-2}^{2}\frac{dy}{(9 + y^2)}$$
$$= \frac{3\rho_s}{2\pi\epsilon_0}\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{y}{3}\right)\Big|_{-2}^{2} = \underline{106\text{ V/m}}$$

The student is encouraged to verify that if the y limits were $-\infty$ to ∞, the result would be that of the infinite charged plane, or $E_x = \rho_s/(2\epsilon_0)$.

b) $P_B(0, 3, 0)$: In this case, $\mathbf{r} = 3\mathbf{a}_y$, and symmetry indicates that only a y component will exist. The integral becomes

$$E_{y,PB} = \frac{\rho_s}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\int_{-2}^{2}\frac{(3 - y)\,dy\,dz}{[(z^2 + 9) - 6y + y^2]^{1.5}} = \frac{\rho_s}{2\pi\epsilon_0}\int_{-2}^{2}\frac{(3 - y)\,dy}{(3 - y)^2}$$
$$= -\frac{\rho_s}{2\pi\epsilon_0}\ln(3 - y)\Big|_{-2}^{2} = \underline{145\text{ V/m}}$$

2.23. Given the surface charge density, $\rho_s = 2\,\mu\text{C/m}^2$, in the region $\rho < 0.2$ m, $z = 0$, and is zero elsewhere, find **E** at:

a) $P_A(\rho = 0, z = 0.5)$: First, we recognize from symmetry that only a z component of **E** will be present. Considering a general point z on the z axis, we have $\mathbf{r} = z\mathbf{a}_z$. Then, with $\mathbf{r}' = \rho\mathbf{a}_\rho$, we obtain $\mathbf{r} - \mathbf{r}' = z\mathbf{a}_z - \rho\mathbf{a}_\rho$. The superposition integral for the z component of **E** will be:

$$E_{z,P_A} = \frac{\rho_s}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{0.2}\frac{z\,\rho\,d\rho\,d\phi}{(\rho^2+z^2)^{1.5}} = -\frac{2\pi\rho_s}{4\pi\epsilon_0}z\left[\frac{1}{\sqrt{z^2+\rho^2}}\right]_0^{0.2}$$
$$= \frac{\rho_s}{2\epsilon_0}z\left[\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{z^2+0.4}}\right]$$

With $z = 0.5$ m, the above evaluates as $E_{z,P_A} = \underline{8.1\text{ kV/m}}$.

b) With z at -0.5 m, we evaluate the expression for E_z to obtain $E_{z,P_B} = \underline{-8.1\text{ kV/m}}$.

2.24. Surface charge density is positioned in free space as follows: $20\,\text{nC/m}^2$ at $x = -3$, $-30\,\text{nC/m}^2$ at $y = 4$, and $40\,\text{nC/m}^2$ at $z = 2$. Find the magnitude of **E** at the three points, $(4, 3, -2)$, $(-2, 5, -1)$, and $(0, 0, 0)$. Since all three sheets are infinite, the field magnitude associated with each one will be $\rho_s/(2\epsilon_0)$, which is position-independent. For this reason, the *net* field magnitude will be the same everywhere, whereas the field direction will depend on which side of a given sheet one is positioned. We take the first point, for example, and find

$$\mathbf{E}_A = \frac{20\times10^{-9}}{2\epsilon_0}\mathbf{a}_x + \frac{30\times10^{-9}}{2\epsilon_0}\mathbf{a}_y - \frac{40\times10^{-9}}{2\epsilon_0}\mathbf{a}_z = 1130\mathbf{a}_x + 1695\mathbf{a}_y - 2260\mathbf{a}_z\text{ V/m}$$

The magnitude of $\mathbf{E}_A$ is thus $\underline{3.04\,\text{kV/m}}$. This will be the magnitude at the other two points as well.

2.25. Find **E** at the origin if the following charge distributions are present in free space: point charge, 12 nC at $P(2, 0, 6)$; uniform line charge density, 3nC/m at $x = -2$, $y = 3$; uniform surface charge density, $0.2\,\text{nC/m}^2$ at $x = 2$. The sum of the fields at the origin from each charge in order is:

$$\mathbf{E} = \left[\frac{(12\times10^{-9})}{4\pi\epsilon_0}\,\frac{(-2\mathbf{a}_x - 6\mathbf{a}_z)}{(4+36)^{1.5}}\right] + \left[\frac{(3\times10^{-9})}{2\pi\epsilon_0}\,\frac{(2\mathbf{a}_x - 3\mathbf{a}_y)}{(4+9)}\right] - \left[\frac{(0.2\times10^{-9})\mathbf{a}_x}{2\epsilon_0}\right]$$
$$= \underline{-3.9\mathbf{a}_x - 12.4\mathbf{a}_y - 2.5\mathbf{a}_z\text{ V/m}}$$

2.26. A uniform line charge density of 5 nC/m is at $y = 0$, $z = 2$ m in free space, while $-5\,\text{nC/m}$ is located at $y = 0$, $z = -2$ m. A uniform surface charge density of $0.3\,\text{nC/m}^2$ is at $y = 0.2$ m, and $-0.3\,\text{nC/m}^2$ is at $y = -0.2$ m. Find $|\mathbf{E}|$ at the origin: Since each pair consists of equal and opposite charges, the effect at the origin is to double the field produce by one of each type. Taking the sum of the fields at the origin from the surface and line charges, respectively, we find:

$$\mathbf{E}(0,0,0) = -2\times\frac{0.3\times10^{-9}}{2\epsilon_0}\mathbf{a}_y - 2\times\frac{5\times10^{-9}}{2\pi\epsilon_0(2)}\mathbf{a}_z = -33.9\mathbf{a}_y - 89.9\mathbf{a}_z$$

so that $|\mathbf{E}| = \underline{96.1\,\text{V/m}}$.

2.27. Given the electric field $\mathbf{E} = (4x - 2y)\mathbf{a}_x - (2x + 4y)\mathbf{a}_y$, find:

a) the equation of the streamline that passes through the point $P(2, 3, -4)$: We write

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-(2x + 4y)}{(4x - 2y)}$$

Thus

$$2(x\,dy + y\,dx) = y\,dy - x\,dx$$

or

$$2\,d(xy) = \frac{1}{2}\,d(y^2) - \frac{1}{2}\,d(x^2)$$

So

$$C_1 + 2xy = \frac{1}{2}y^2 - \frac{1}{2}x^2$$

or

$$y^2 - x^2 = 4xy + C_2$$

Evaluating at $P(2, 3, -4)$, obtain:

$$9 - 4 = 24 + C_2, \text{ or } C_2 = -19$$

Finally, at P, the requested equation is

$$\underline{y^2 - x^2 = 4xy - 19}$$

b) a unit vector specifying the direction of $\mathbf{E}$ at $Q(3, -2, 5)$: Have $\mathbf{E}_Q = [4(3) + 2(2)]\mathbf{a}_x - [2(3) - 4(2)]\mathbf{a}_y = 16\mathbf{a}_x + 2\mathbf{a}_y$. Then $|\mathbf{E}| = \sqrt{16^2 + 4} = 16.12$ So

$$\mathbf{a}_Q = \frac{16\mathbf{a}_x + 2\mathbf{a}_y}{16.12} = \underline{0.99\mathbf{a}_x + 0.12\mathbf{a}_y}$$

2.28. Let $\mathbf{E} = 5x^3\,\mathbf{a}_x - 15x^2y\,\mathbf{a}_y$, and find:

a) the equation of the streamline that passes through $P(4, 2, 1)$: Write

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-15x^2y}{5x^3} = \frac{-3y}{x}$$

So

$$\frac{dy}{y} = -3\,\frac{dx}{x} \quad\Rightarrow\quad \ln y = -3\ln x + \ln C$$

Thus

$$y = e^{-3\ln x}e^{\ln C} = \frac{C}{x^3}$$

At P, have $2 = C/(4)^3 \Rightarrow C = 128$. Finally, at P,

$$\underline{y = \frac{128}{x^3}}$$

2.28. (continued)

b) a unit vector $\mathbf{a}_E$ specifying the direction of $\mathbf{E}$ at $Q(3,-2,5)$: At Q, $\mathbf{E}_Q = 135\mathbf{a}_x + 270\mathbf{a}_y$, and $|\mathbf{E}_Q| = 301.9$. Thus $\mathbf{a}_E = \underline{0.45\mathbf{a}_x + 0.89\mathbf{a}_y}$.

c) a unit vector $\mathbf{a}_N = (l, m, 0)$ that is perpendicular to $\mathbf{a}_E$ at Q: Since this vector is to have no z component, we can find it through $\mathbf{a}_N = \pm(\mathbf{a}_E \times \mathbf{a}_z)$. Performing this, we find $\mathbf{a}_N = \underline{\pm(0.89\mathbf{a}_x - 0.45\mathbf{a}_y)}$.

2.29. If $\mathbf{E} = 20e^{-5y}\left(\cos 5x\,\mathbf{a}_x - \sin 5x\,\mathbf{a}_y\right)$, find:

a) $|\mathbf{E}|$ at $P(\pi/6, 0.1, 2)$: Substituting this point, we obtain $\mathbf{E}_P = -10.6\mathbf{a}_x - 6.1\mathbf{a}_y$, and so $|\mathbf{E}_P| = \underline{12.2}$.

b) a unit vector in the direction of $\mathbf{E}_P$: The unit vector associated with $\mathbf{E}$ is just $\left(\cos 5x\,\mathbf{a}_x - \sin 5x\,\mathbf{a}_y\right)$, which evaluated at P becomes $\mathbf{a}_E = \underline{-0.87\mathbf{a}_x - 0.50\mathbf{a}_y}$.

c) the equation of the direction line passing through P: Use

$$\frac{dy}{dx} = \frac{-\sin 5x}{\cos 5x} = -\tan 5x \quad \Rightarrow \quad dy = -\tan 5x\,dx$$

Thus $y = \frac{1}{5}\ln\cos 5x + C$. Evaluating at P, we find $C = 0.13$, and so

$$\underline{y = \frac{1}{5}\ln\cos 5x + 0.13}$$

2.30. Given the electric field intensity $\mathbf{E} = 400y\mathbf{a}_x + 400x\mathbf{a}_y$ V/m, find:

a) the equation of the streamline passing through the point $A(2, 1, -2)$: Write:

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{x}{y} \quad \Rightarrow \quad x\,dx = y\,dy$$

Thus $x^2 = y^2 + C$. Evaluating at A yields $C = 3$, so the equation becomes

$$\underline{\frac{x^2}{3} - \frac{y^2}{3} = 1}$$

b) the equation of the surface on which $|\mathbf{E}| = 800$ V/m: Have $|\mathbf{E}| = 400\sqrt{x^2+y^2} = 800$. Thus $\underline{x^2 + y^2 = 4}$, or we have a circular-cylindrical surface, centered on the z axis, and of radius 2.

c) A sketch of the part a equation would yield a parabola, centered at the origin, whose axis is the positive x axis, and for which the slopes of the asymptotes are ± 1.

d) A sketch of the trace produced by the intersection of the surface of part b with the $z = 0$ plane would yield a circle centered at the origin, of radius 2.

2.31. In cylindrical coordinates with $\mathbf{E}(\rho, \phi) = E_\rho(\rho, \phi)\mathbf{a}_\rho + E_\phi(\rho, \phi)\mathbf{a}_\phi$, the differential equation describing the direction lines is $E_\rho/E_\phi = d\rho/(\rho d\phi)$ in any constant-z plane. Derive the equation of the line passing through the point $P(\rho = 4, \phi = 10^\circ, z = 2)$ in the field $\mathbf{E} = 2\rho^2 \cos 3\phi \mathbf{a}_\rho + 2\rho^2 \sin 3\phi \mathbf{a}_\phi$: Using the given information, we write

$$\frac{E_\rho}{E_\phi} = \frac{d\rho}{\rho d\phi} = \cot 3\phi$$

Thus

$$\frac{d\rho}{\rho} = \cot 3\phi \, d\phi \quad \Rightarrow \quad \ln \rho = \frac{1}{3} \ln \sin 3\phi + \ln C$$

or $\rho = C(\sin 3\phi)^{1/3}$. Evaluate this at P to obtain $C = 7.14$. Finally,

$$\underline{\rho^3 = 364 \sin 3\phi}$$