CHAPTER 3

3.1. An empty metal paint can is placed on a marble table, the lid is removed, and both parts are discharged (honorably) by touching them to ground. An insulating nylon thread is glued to the center of the lid, and a penny, a nickel, and a dime are glued to the thread so that they are not touching each other. The penny is given a charge of +5 nC, and the nickel and dime are discharged. The assembly is lowered into the can so that the coins hang clear of all walls, and the lid is secured. The outside of the can is again touched momentarily to ground. The device is carefully disassembled with insulating gloves and tools.

a) What charges are found on each of the five metallic pieces? All coins were insulated during the entire procedure, so they will retain their original charges: Penny: $\underline{+5}$ nC; nickel: $\underline{0}$; dime: $\underline{0}$. The penny's charge will have induced an equal and opposite negative charge (-5 nC) on the inside wall of the can and lid. This left a charge layer of +5 nC on the outside surface which was neutralized by the ground connection. Therefore, the can retained a net charge of $\underline{-5\,\text{nC}}$ after disassembly.

b) If the penny had been given a charge of $+5$ nC, the dime a charge of -2 nC, and the nickel a charge of -1 nC, what would the final charge arrangement have been? Again, since the coins are insulated, they retain their original charges. The charge induced on the inside wall of the can and lid is equal to negative the sum of the coin charges, or -2 nC. This is the charge that the can/lid contraption retains after grounding and disassembly.

3.2. A point charge of 12 nC is located at the origin. four uniform line charges are located in the $x = 0$ plane as follows: 80 nC/m at $y = -1$ and -5 m, -50 nC/m at $y = -2$ and -4 m.

a) Find **D** at $P(0, -3, 2)$: Note that this point lies in the center of a symmetric arrangement of line charges, whose fields will all cancel at that point. Thus **D** arise from the point charge alone, and will be

$$\mathbf{D} = \frac{12 \times 10^{-9}(-3\mathbf{a}_y + 2\mathbf{a}_z)}{4\pi(3^2 + 2^2)^{1.5}} = -6.11 \times 10^{-11}\mathbf{a}_y + 4.07 \times 10^{-11}\mathbf{a}_z \ \text{C/m}^2$$
$$= \underline{-61.1\mathbf{a}_y + 40.7\mathbf{a}_z \ \text{pC/m}^2}$$

b) How much electric flux crosses the plane $y = -3$ and in what direction? The plane intercepts all flux that enters the $-y$ half-space, or exactly half the total flux of 12 nC. The answer is thus $\underline{6\text{ nC}}$ and in the $-\mathbf{a}_y$ direction.

c) How much electric flux leaves the surface of a sphere, 4m in radius, centered at $C(0, -3, 0)$? This sphere encloses the point charge, so its flux of 12 nC is included. The line charge contributions are most easily found by translating the whole assembly (sphere and line charges) such that the sphere is centered at the origin, with line charges now at $y = \pm 1$ and ± 2. The flux from the line charges will equal the total line charge that lies within the sphere. The length of each of the inner two line charges (at $y = \pm 1$) will be

$$h_1 = 2r\cos\theta_1 = 2(4)\cos\left[\sin^{-1}\left(\frac{1}{4}\right)\right] = 1.94\,\text{m}$$

That of each of the outer two line charges (at $y = \pm 2$) will be

$$h_2 = 2r\cos\theta_2 = 2(4)\cos\left[\sin^{-1}\left(\frac{2}{4}\right)\right] = 1.73\,\text{m}$$

3.2c. (continued) The total charge enclosed in the sphere (and the outward flux from it) is now

$$Q_l + Q_p = 2(1.94)(-50 \times 10^{-9}) + 2(1.73)(80 \times 10^{-9}) + 12 \times 10^{-9} = \underline{348\,\mathrm{nC}}$$

3.3. The cylindrical surface $\rho = 8$ cm contains the surface charge density, $\rho_s = 5e^{-20|z|}$ nC/m^2.

a) What is the total amount of charge present? We integrate over the surface to find:

$$Q = 2\int_0^\infty \int_0^{2\pi} 5e^{-20z}(.08)d\phi\, dz \text{ nC} = 20\pi(.08)\left(\frac{-1}{20}\right)e^{-20z}\Big|_0^\infty = \underline{0.25\,\mathrm{nC}}$$

b) How much flux leaves the surface $\rho = 8$ cm, 1 cm $< z <$ 5cm, $30^\circ < \phi < 90^\circ$? We just integrate the charge density on that surface to find the flux that leaves it.

$$\Phi = Q' = \int_{.01}^{.05}\int_{30^\circ}^{90^\circ} 5e^{-20z}(.08)\,d\phi\,dz \text{ nC} = \left(\frac{90-30}{360}\right)2\pi(5)(.08)\left(\frac{-1}{20}\right)e^{-20z}\Big|_{.01}^{.05}$$
$$= 9.45 \times 10^{-3}\,\mathrm{nC} = \underline{9.45\,\mathrm{pC}}$$

3.4. The cylindrical surfaces $\rho = 1, 2,$ and 3 cm carry uniform surface charge densities of 20, -8, and 5 nC/m^2, respectively.

a) How much electric flux passes through the closed surface $\rho = 5$ cm, $0 < z < 1$ m? Since the densities are uniform, the flux will be

$$\Phi = 2\pi(a\rho_{s1} + b\rho_{s2} + c\rho_{s3})(1\,\mathrm{m}) = 2\pi\,[(.01)(20) - (.02)(8) + (.03)(5)] \times 10^{-9} = \underline{1.2\,\mathrm{nC}}$$

b) Find **D** at $P(1\,\mathrm{cm}, 2\,\mathrm{cm}, 3\,\mathrm{cm})$: This point lies at radius $\sqrt{5}$ cm, and is thus inside the outermost charge layer. This layer, being of uniform density, will not contribute to **D** at P. We know that in cylindrical coordinates, the layers at 1 and 2 cm will produce the flux density:

$$\mathbf{D} = D_\rho \mathbf{a}_\rho = \frac{a\rho_{s1} + b\rho_{s2}}{\rho}\mathbf{a}_\rho$$

or

$$D_\rho = \frac{(.01)(20) + (.02)(-8)}{\sqrt{.05}} = 1.8\,\mathrm{nC/m^2}$$

At P, $\phi = \tan^{-1}(2/1) = 63.4^\circ$. Thus $D_x = 1.8\cos\phi = 0.8$ and $D_y = 1.8\sin\phi = 1.6$. Finally,

$$\mathbf{D}_P = \underline{(0.8\mathbf{a}_x + 1.6\mathbf{a}_y)\ \mathrm{nC/m^2}}$$

3.5. Let $\mathbf{D} = 4xy\mathbf{a}_x + 2(x^2 + z^2)\mathbf{a}_y + 4yz\mathbf{a}_z$ C/m^2 and evaluate surface integrals to find the total charge enclosed in the rectangular parallelepiped $0 < x < 2$, $0 < y < 3$, $0 < z < 5$ m: Of the 6 surfaces to consider, only 2 will contribute to the net outward flux. Why? First consider the planes at $y = 0$ and 3. The y component of $\mathbf{D}$ will penetrate those surfaces, but will be inward at $y = 0$ and outward at $y = 3$, while having the same magnitude in both cases. These fluxes will thus cancel. At the $x = 0$ plane, $D_x = 0$ and at the $z = 0$ plane, $D_z = 0$, so there will be no flux contributions from these surfaces. This leaves the 2 remaining surfaces at $x = 2$ and $z = 5$. The net outward flux becomes:

$$\Phi = \int_0^5 \int_0^3 \mathbf{D}\Big|_{x=2} \cdot \mathbf{a}_x \, dy\, dz + \int_0^3 \int_0^2 \mathbf{D}\Big|_{z=5} \cdot \mathbf{a}_z \, dx\, dy$$
$$= 5 \int_0^3 4(2)y\, dy \; + \; 2 \int_0^3 4(5)y\, dy = \underline{360 \text{ C}}$$

3.6. Two uniform line charges, each 20 nC/m, are located at $y = 1$, $z = \pm 1$ m. Find the total flux leaving a sphere of radius 2 m if it is centered at

a) $A(3, 1, 0)$: The result will be the same if we move the sphere to the origin and the line charges to $(0, 0, \pm 1)$. The length of the line charge within the sphere is given by $l = 4\sin[\cos^{-1}(1/2)] = 3.46$. With two line charges, symmetrically arranged, the total charge enclosed is given by $Q = 2(3.46)(20\,\text{nC/m}) = \underline{139\,\text{nC}}$

b) $B(3, 2, 0)$: In this case the result will be the same if we move the sphere to the origin and keep the charges where they were. The length of the line joining the origin to the midpoint of the line charge (in the yz plane) is $l_1 = \sqrt{2}$. The length of the line joining the origin to either endpoint of the line charge is then just the sphere radius, or 2. The half-angle subtended at the origin by the line charge is then $\psi = \cos^{-1}(\sqrt{2}/2) = 45^\circ$. The length of each line charge in the sphere is then $l_2 = 2 \times 2\sin\psi = 2\sqrt{2}$. The total charge enclosed (with two line charges) is now $Q' = 2(2\sqrt{2})(20\,\text{nC/m}) = \underline{113\,\text{nC}}$

3.7. Volume charge density is located in free space as $\rho_v = 2e^{-1000r}$ nC/m^3 for $0 < r < 1$ mm, and $\rho_v = 0$ elsewhere.

a) Find the total charge enclosed by the spherical surface $r = 1$ mm: To find the charge we integrate:

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{.001} 2e^{-1000r} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

Integration over the angles gives a factor of 4π. The radial integration we evaluate using tables; we obtain

$$Q = 8\pi \left[\frac{-r^2 e^{-1000r}}{1000}\Big|_0^{.001} + \frac{2}{1000} \frac{e^{-1000r}}{(1000)^2}(-1000r - 1)\Big|_0^{.001} \right] = \underline{4.0 \times 10^{-9}\,\text{nC}}$$

b) By using Gauss's law, calculate the value of D_r on the surface $r = 1$ mm: The gaussian surface is a spherical shell of radius 1 mm. The enclosed charge is the result of part a. We thus write $4\pi r^2 D_r = Q$, or

$$D_r = \frac{Q}{4\pi r^2} = \frac{4.0 \times 10^{-9}}{4\pi(.001)^2} = \underline{3.2 \times 10^{-4}\,\text{nC/m}^2}$$

3.8. Uniform line charges of 5 nC/m ar located in free space at $x = 1$, $z = 1$, and at $y = 1$, $z = 0$.

a) Obtain an expression for **D** in cartesian coordinates at $P(0, 0, z)$. In general, we have

$$\mathbf{D}(z) = \frac{\rho_s}{2\pi}\left[\frac{\mathbf{r}_1 - \mathbf{r}'_1}{|\mathbf{r}_1 - \mathbf{r}'_1|^2} + \frac{\mathbf{r}_2 - \mathbf{r}'_2}{|\mathbf{r}_2 - \mathbf{r}'_2|^2}\right]$$

where $\mathbf{r}_1 = \mathbf{r}_2 = z\mathbf{a}_z$, $\mathbf{r}'_1 = \mathbf{a}_y$, and $\mathbf{r}'_2 = \mathbf{a}_x + \mathbf{a}_z$. Thus

$$\mathbf{D}(z) = \frac{\rho_s}{2\pi}\left[\frac{[z\mathbf{a}_z - \mathbf{a}_y]}{[1 + z^2]} + \frac{[(z-1)\mathbf{a}_z - \mathbf{a}_x]}{[1 + (z-1)^2]}\right]$$
$$= \underline{\frac{\rho_s}{2\pi}\left[\frac{-\mathbf{a}_x}{[1 + (z-1)^2]} - \frac{\mathbf{a}_y}{[1 + z^2]} + \left(\frac{(z-1)}{[1 + (z-1)^2]} + \frac{z}{[1 + z^2]}\right)\mathbf{a}_z\right]}$$

b) Plot $|\mathbf{D}|$ vs. z at P, $-3 < z < 10$: Using part a, we find the magnitude of **D** to be

$$|\mathbf{D}| = \frac{\rho_s}{2\pi}\left[\frac{1}{[1 + (z-1)^2]^2} + \frac{1}{[1 + z^2]^2} + \left(\frac{(z-1)}{[1 + (z-1)^2]} + \frac{z}{[1 + z^2]}\right)^2\right]^{1/2}$$

A plot of this over the specified range is shown in Prob3.8.pdf.

3.9. A uniform volume charge density of $80\,\mu\text{C/m}^3$ is present throughout the region $8\,\text{mm} < r < 10\,\text{mm}$. Let $\rho_v = 0$ for $0 < r < 8\,\text{mm}$.

a) Find the total charge inside the spherical surface $r = 10$ mm: This will be

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_{.008}^{.010} (80 \times 10^{-6})r^2 \sin\theta\, dr\, d\theta\, d\phi = 4\pi \times (80 \times 10^{-6})\frac{r^3}{3}\bigg|_{.008}^{.010}$$
$$= 1.64 \times 10^{-10}\,\text{C} = \underline{164\,\text{pC}}$$

b) Find D_r at $r = 10$ mm: Using a spherical gaussian surface at $r = 10$, Gauss' law is written as $4\pi r^2 D_r = Q = 164 \times 10^{-12}$, or

$$D_r(10\,\text{mm}) = \frac{164 \times 10^{-12}}{4\pi(.01)^2} = 1.30 \times 10^{-7}\,\text{C/m}^2 = \underline{130\,\text{nC/m}^2}$$

c) If there is no charge for $r > 10$ mm, find D_r at $r = 20$ mm: This will be the same computation as in part b, except the gaussian surface now lies at 20 mm. Thus

$$D_r(20\,\text{mm}) = \frac{164 \times 10^{-12}}{4\pi(.02)^2} = 3.25 \times 10^{-8}\,\text{C/m}^2 = \underline{32.5\,\text{nC/m}^2}$$

3.10. Let $\rho_s = 8\,\mu\text{C/m}^2$ in the region where $x = 0$ and $-4 < z < 4$ m, and let $\rho_s = 0$ elsewhere. Find **D** at $P(x, 0, z)$, where $x > 0$: The sheet charge can be thought of as an assembly of infinitely-long parallel strips that lie parallel to the y axis in the yz plane, and where each is of thickness dz. The field from each strip is that of an infinite line charge, and so we can construct the field at P from a single strip as:

$$d\mathbf{D}_P = \frac{\rho_s\, dz'}{2\pi}\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2}$$

3.10 (continued) where $\mathbf{r} = x\mathbf{a}_x + z\mathbf{a}_z$ and $\mathbf{r}' = z'\mathbf{a}_z$ We distinguish between the fixed coordinate of P, z, and the variable coordinate, z', that determines the location of each charge strip. To find the net field at P, we sum the contributions of each strip by integrating over z':

$$\mathbf{D}_P = \int_{-4}^{4} \frac{8 \times 10^{-6}\, dz'\,(x\mathbf{a}_x + (z - z')\mathbf{a}_z)}{2\pi[x^2 + (z - z')^2]}$$

We can re-arrange this to determine the integral forms:

$$\mathbf{D}_P = \frac{8 \times 10^{-6}}{2\pi}\left[(x\mathbf{a}_x + z\mathbf{a}_z)\int_{-4}^{4} \frac{dz'}{(x^2 + z^2) - 2zz' + (z')^2} - \mathbf{a}_z \int_{-4}^{4} \frac{z'\,dz'}{(x^2 + z^2) - 2zz' + (z')^2}\right]$$

Using integral tables, we find

$$\mathbf{D}_P = \frac{4 \times 10^{-6}}{\pi}\left[(x\mathbf{a}_x + z\mathbf{a}_z)\frac{1}{x}\tan^{-1}\left(\frac{2z' - 2z}{2x}\right) - \left[\frac{1}{2}\ln(x^2 + z^2 - 2zz' + (z')^2) + \frac{2z}{2}\frac{1}{x}\tan^{-1}\left(\frac{2z' - 2z}{2x}\right)\right]\mathbf{a}_z\right]_{-4}^{4}$$

which evaluates as

$$\mathbf{D}_P = \frac{4 \times 10^{-6}}{\pi}\left\{\left[\tan^{-1}\left(\frac{z+4}{x}\right) - \tan^{-1}\left(\frac{z-4}{x}\right)\right]\mathbf{a}_x + \frac{1}{2}\ln\left[\frac{x^2 + (z+4)^2}{x^2 + (z-4)^2}\right]\mathbf{a}_z\right\} \ \ \mathrm{C/m^2}$$

The student is invited to verify that for very small x or for a very large sheet (allowing z' to approach infinity), the above expression reduces to the expected form, $\mathbf{D}_P = \rho_s/2$. Note also that the expression is valid for all x (positive or negative values).

3.11. In cylindrical coordinates, let $\rho_v = 0$ for $\rho < 1$ mm, $\rho_v = 2\sin(2000\pi\rho)$ nC/m^3 for $1\,\mathrm{mm} < \rho < 1.5\,\mathrm{mm}$, and $\rho_v = 0$ for $\rho > 1.5\,\mathrm{mm}$. Find **D** everywhere: Since the charge varies only with radius, and is in the form of a cylinder, symmetry tells us that the flux density will be radially-directed and will be constant over a cylindrical surface of a fixed radius. Gauss' law applied to such a surface of unit length in z gives:

a) for $\rho < 1$ mm, $\underline{D_\rho = 0}$, since no charge is enclosed by a cylindrical surface whose radius lies within this range.

b) for $1\,\mathrm{mm} < \rho < 1.5\,\mathrm{mm}$, we have

$$2\pi\rho D_\rho = 2\pi\int_{.001}^{\rho} 2 \times 10^{-9}\sin(2000\pi\rho')\rho'\,d\rho' = 4\pi \times 10^{-9}\left[\frac{1}{(2000\pi)^2}\sin(2000\pi\rho) - \frac{\rho}{2000\pi}\cos(2000\pi\rho)\right]_{.001}^{\rho}$$

or finally,

$$D_\rho = \frac{10^{-15}}{2\pi^2\rho}\left[\sin(2000\pi\rho) + 2\pi\left[1 - 10^3\rho\cos(2000\pi\rho)\right]\right] \mathrm{C/m^2} \quad (1\,\mathrm{mm} < \rho < 1.5\,\mathrm{mm})$$

3.11. (continued)

c) for $\rho > 1.5\,\text{mm}$, the gaussian cylinder now lies at radius ρ *outside* the charge distribution, so the integral that evaluates the enclosed charge now includes the entire charge distribution. To accomplish this, we change the upper limit of the integral of part b from ρ to 1.5 mm, finally obtaining:

$$D_\rho = \frac{2.5 \times 10^{-15}}{\pi\rho}\ \text{C/m}^2 \quad (\rho > 1.5\,\text{mm})$$

3.12. A nonuniform volume charge density, $\rho_v = 120r\ \text{C/m}^3$, lies within the spherical surface $r = 1\,\text{m}$, and $\rho_v = 0$ everywhere else.

a) Find D_r everywhere. For $r < 1$ m, we apply Gauss' law to a spherical surface of radius r within this range to find

$$4\pi r^2 D_r = 4\pi \int_0^r 120r'(r')^2\,dr' = 120\pi r^4$$

Thus $D_r = \underline{(30r^2)}$ for $r < 1$ m. For $r > 1$ m, the gaussian surface lies outside the charge distribution. The set up is the same, except the upper limit of the above integral is 1 instead of r. This results in $D_r = \underline{(30/r^2)}$ for $r > 1$ m.

b) What surface charge density, ρ_{s2}, should be on the surface $r = 2$ such that $D_{r,r=2-} = 2D_{r,r=2+}$? At $r = 2^-$, we have $D_{r,r=2-} = 30/2^2 = 15/2$, from part a. The flux density in the region $r > 2$ arising from a surface charge at $r = 2$ is found from Gauss' law through

$$4\pi r^2 D_{rs} = 4\pi(2)^2\rho_{s2} \;\Rightarrow\; D_{rs} = \frac{4\rho_{s2}}{r^2}$$

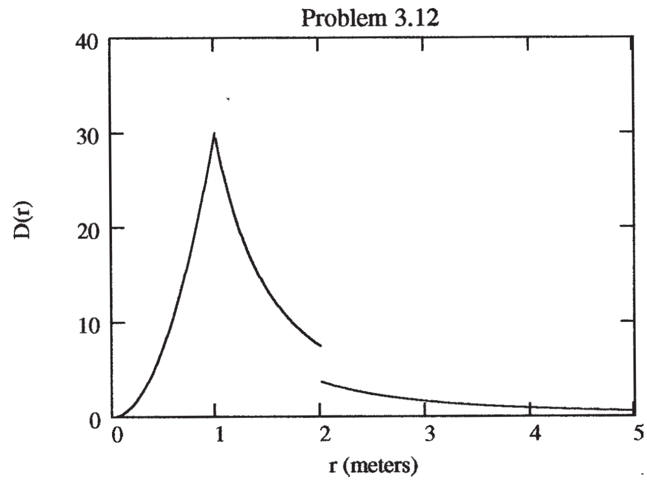
The total flux density in the region $r > 2$ arising from the two distributions is

$$D_{rT} = \frac{30}{r^2} + \frac{4\rho_{s2}}{r^2}$$

Our requirement that $D_{r,r=2-} = 2D_{r,r=2+}$ becomes

$$\frac{30}{2^2} = 2\left(\frac{30}{2^2} + \rho_{s2}\right) \;\Rightarrow\; \rho_{s2} = \underline{-\frac{15}{4}\ \text{C/m}^2}$$

c) Make a sketch of D_r vs. r for $0 < r < 5$ m with both distributions present. With both charges, $D_r(r < 1) = 30r^2$, $D_r(1 < r < 2) = 30/r^2$, and $D_r(r > 2) = 15/r^2$. These are plotted on the next page.



3.13. Spherical surfaces at $r = 2,\ 4,$ and 6 m carry uniform surface charge densities of $20\,\text{nC/m}^2$, $-4\,\text{nC/m}^2$, and ρ_{s0}, respectively.

a) Find **D** at $r = 1,\ 3$ and 5 m: Noting that the charges are spherically-symmetric, we ascertain that **D** will be radially-directed and will vary only with radius. Thus, we apply Gauss' law to spherical shells in the following regions: $r < 2$: Here, no charge is enclosed, and so $\underline{D_r = 0}$.

$$2 < r < 4: \quad 4\pi r^2 D_r = 4\pi(2)^2(20\times 10^{-9}) \;\Rightarrow\; D_r = \frac{80\times 10^{-9}}{r^2}\ \text{C/m}^2$$

So $D_r(r=3) = \underline{8.9\times 10^{-9}\,\text{C/m}^2}$.

$$4 < r < 6: \quad 4\pi r^2 D_r = 4\pi(2)^2(20\times 10^{-9}) + 4\pi(4)^2(-4\times 10^{-9}) \;\Rightarrow\; D_r = \frac{16\times 10^{-9}}{r^2}$$

So $D_r(r=5) = \underline{6.4\times 10^{-10}\,\text{C/m}^2}$.

b) Determine ρ_{s0} such that $\mathbf{D} = 0$ at $r = 7$ m. Since fields will decrease as $1/r^2$, the question could be re-phrased to ask for ρ_{s0} such that $\mathbf{D} = 0$ at *all* points where $r > 6$ m. In this region, the total field will be

$$D_r(r > 6) = \frac{16\times 10^{-9}}{r^2} + \frac{\rho_{s0}(6)^2}{r^2}$$

Requiring this to be zero, we find $\rho_{s0} = \underline{-(4/9)\times 10^{-9}\,\text{C/m}^2}$.

3.14. If $\rho_v = 5\,\text{nC/m}^3$ for $0 < \rho < 1$ mm and no other charges are present:

a) find D_ρ for $\rho < 1$ mm: Applying Gauss' law to a cylindrical surface of unit length in z, and of radius $\rho < 1$ mm, we find

$$2\pi\rho D_\rho = \pi\rho^2(5\times 10^{-9}) \;\Rightarrow\; D_\rho = \underline{2.5\,\rho\times 10^{-9}\,\text{C/m}^2}$$

3.14b. find D_ρ for $\rho > 1$ mm: The Gaussian cylinder now lies outside the charge, so

$$2\pi\rho D_\rho = \pi(.001)^2(5\times10^{-9}) \quad\Rightarrow\quad D_\rho = \underline{\frac{2.5\times10^{-15}}{\rho}\ \text{C/m}^2}$$

c) What line charge ρ_L at $\rho = 0$ would give the same result for part b? The line charge field will be

$$D_r = \frac{\rho_L}{2\pi\rho} = \frac{2.5\times10^{-15}}{\rho} \quad (\text{part } b)$$

Thus $\rho_L = \underline{5\pi\times10^{-15}\ \text{C/m}}$. In all answers, ρ is expressed in meters.

3.15. Volume charge density is located as follows: $\rho_v = 0$ for $\rho < 1$ mm and for $\rho > 2$ mm, $\rho_v = 4\rho\ \mu\text{C/m}^3$ for $1 < \rho < 2$ mm.

a) Calculate the total charge in the region $0 < \rho < \rho_1$, $0 < z < L$, where $1 < \rho_1 < 2$ mm: We find

$$Q = \int_0^L \int_0^{2\pi} \int_{.001}^{\rho_1} 4\rho\,\rho\,d\rho\,d\phi\,dz = \underline{\frac{8\pi L}{3}[\rho_1^3 - 10^{-9}]\ \mu\text{C}}$$

where ρ_1 is in meters.

b) Use Gauss' law to determine D_ρ at $\rho = \rho_1$: Gauss' law states that $2\pi\rho_1 L D_\rho = Q$, where Q is the result of part a. Thus

$$D_\rho(\rho_1) = \underline{\frac{4(\rho_1^3 - 10^{-9})}{3\rho_1}\ \mu\text{C/m}^2}$$

where ρ_1 is in meters.

c) Evaluate D_ρ at $\rho = 0.8$ mm, 1.6 mm, and 2.4 mm: At $\rho = 0.8$ mm, no charge is enclosed by a cylindrical gaussian surface of that radius, so $D_\rho(0.8\text{mm}) = \underline{0}$. At $\rho = 1.6$ mm, we evaluate the part b result at $\rho_1 = 1.6$ to obtain:

$$D_\rho(1.6\text{mm}) = \frac{4[(.0016)^3 - (.0010)^3]}{3(.0016)} = \underline{3.6\times10^{-6}\ \mu\text{C/m}^2}$$

At $\rho = 2.4$, we evaluate the charge integral of part a from .001 to .002, and Gauss' law is written as

$$2\pi\rho L D_\rho = \frac{8\pi L}{3}[(.002)^2 - (.001)^2]\ \mu\text{C}$$

from which $D_\rho(2.4\text{mm}) = \underline{3.9\times10^{-6}\ \mu\text{C/m}^2}$.

3.16. Given the electric flux density, $\mathbf{D} = 2xy\,\mathbf{a}_x + x^2\,\mathbf{a}_y + 6z^3\,\mathbf{a}_z$ C/m^2:

a) use Gauss' law to evaluate the total charge enclosed in the volume $0 < x, y, z < a$: We call the surfaces at $x = a$ and $x = 0$ the front and back surfaces respectively, those at $y = a$ and $y = 0$ the right and left surfaces, and those at $z = a$ and $z = 0$ the top and bottom surfaces. To evaluate the total charge, we integrate $\mathbf{D}\cdot\mathbf{n}$ over all six surfaces and sum the results:

$$\Phi = Q = \oint \mathbf{D}\cdot\mathbf{n}\,da = \underbrace{\int_0^a\int_0^a 2ay\,dy\,dz}_{\text{front}} + \underbrace{\int_0^a\int_0^a -2(0)y\,dy\,dz}_{\text{back}}$$

$$+ \underbrace{\int_0^a\int_0^a -x^2\,dx\,dz}_{\text{left}} + \underbrace{\int_0^a\int_0^a x^2\,dx\,dz}_{\text{right}} + \underbrace{\int_0^a\int_0^a -6(0)^3\,dx\,dy}_{\text{bottom}} + \underbrace{\int_0^a\int_0^a 6a^3\,dx\,dy}_{\text{top}}$$

3.16a. (continued) Noting that the back and bottom integrals are zero, and that the left and right integrals cancel, we evaluate the remaining two (front and top) to obtain $Q = \underline{6a^5 + a^4}$.

b) use Eq. (8) to find an approximate value for the above charge. Evaluate the derivatives at $P(a/2, a/2, a/2)$: In this application, Eq. (8) states that $Q \doteq (\nabla \cdot \mathbf{D}\big|_P)\Delta v$. We find $\nabla \cdot \mathbf{D} = 2x + 18z^2$, which when evaluated at P becomes $\nabla \cdot \mathbf{D}\big|_P = a + 4.5a^2$. Thus $Q \doteq (a + 4.5a^2)a^3 = \underline{4.5a^5 + a^4}$

c) Show that the results of parts a and b agree in the limit as $a \to 0$. In this limit, both expressions reduce to $Q = a^4$, and so they agree.

3.17. A cube is defined by $1 < x, y, z < 1.2$. If $\mathbf{D} = 2x^2y\mathbf{a}_x + 3x^2y^2\mathbf{a}_y$ C/m^2:

a) apply Gauss' law to find the total flux leaving the closed surface of the cube. We call the surfaces at $x = 1.2$ and $x = 1$ the front and back surfaces respectively, those at $y = 1.2$ and $y = 1$ the right and left surfaces, and those at $z = 1.2$ and $z = 1$ the top and bottom surfaces. To evaluate the total charge, we integrate $\mathbf{D} \cdot \mathbf{n}$ over all six surfaces and sum the results. We note that there is no z component of $\mathbf{D}$, so there will be no outward flux contributions from the top and bottom surfaces. The fluxes through the remaining four are

$$\Phi = Q = \oint \mathbf{D} \cdot \mathbf{n}\, da = \underbrace{\int_1^{1.2}\int_1^{1.2} 2(1.2)^2 y\, dy\, dz}_{\text{front}} + \underbrace{\int_1^{1.2}\int_1^{1.2} -2(1)^2 y\, dy\, dz}_{\text{back}}$$

$$+ \underbrace{\int_1^{1.2}\int_1^{1.2} -3x^2(1)^2\, dx\, dz}_{\text{left}} + \underbrace{\int_1^{1.2}\int_1^{1.2} 3x^2(1.2)^2\, dx\, dz}_{\text{right}} = \underline{0.1028\,\text{C}}$$

b) evaluate $\nabla \cdot \mathbf{D}$ at the center of the cube: This is

$$\nabla \cdot \mathbf{D} = \Big[4xy + 6x^2y\Big]_{(1.1,1.1)} = 4(1.1)^2 + 6(1.1)^3 = \underline{12.83}$$

c) Estimate the total charge enclosed within the cube by using Eq. (8): This is

$$Q \doteq \nabla \cdot \mathbf{D}\Big|_{\text{center}} \times \Delta v = 12.83 \times (0.2)^3 = \underline{0.1026} \;\; \text{Close!}$$

3.18. Let a vector field by given by $\mathbf{G} = 5x^4y^4z^4\,\mathbf{a}_y$. Evaluate both sides of Eq. (8) for this $\mathbf{G}$ field and the volume defined by $x = 3$ and 3.1, $y = 1$ and 1.1, and $z = 2$ and 2.1. Evaluate the partial derivatives at the center of the volume. First find

$$\nabla \cdot \mathbf{G} = \frac{\partial G_y}{\partial y} = 20x^4y^3z^4$$

The center of the cube is located at (3.05,1.05,2.05), and the volume is $\Delta v = (0.1)^3 = 0.001$. Eq. (8) then becomes

$$\Phi \doteq 20(3.05)^4(1.05)^3(2.05)^4(0.001) = \underline{35.4}$$

3.19. A spherical surface of radius 3 mm is centered at $P(4, 1, 5)$ in free space. Let $\mathbf{D} = x\mathbf{a}_x$ C/m^2. Use the results of Sec. 3.4 to estimate the net electric flux leaving the spherical surface: We use $\Phi \doteq \nabla \cdot \mathbf{D}\Delta v$, where in this case $\nabla \cdot \mathbf{D} = (\partial/\partial x)x = 1$ C/m^3. Thus

$$\Phi \doteq \frac{4}{3}\pi(.003)^3(1) = 1.13 \times 10^{-7}\,\text{C} = \underline{113\,\text{nC}}$$

3.20. A cube of volume a^3 has its faces parallel to the cartesian coordinate surfaces. It is centered at $P(3, -2, 4)$. Given the field $\mathbf{D} = 2x^3\mathbf{a}_x$ C/m^2:

a) calculate div $\mathbf{D}$ at P: In the present case, this will be

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} = \frac{dD_x}{dx} = \underline{54\,\text{C/m}^3}$$

b) evaluate the fraction in the rightmost side of Eq. (13) for $a = 1$ m, 0.1 m, and 1 mm: With the field having only an x component, flux will pentrate only the two surfaces at $x = 3 \pm a/2$, each of which has surface area a^2. The cube volume is $\Delta v = a^3$. The equation reads:

$$\frac{\oint \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \frac{1}{a^3}\left[2\left(3 + \frac{a}{2}\right)^3 a^2 - 2\left(3 - \frac{a}{2}\right)^3 a^2\right] = \frac{2}{a}\left[(3 + \frac{a}{2})^3 - (3 - \frac{a}{2})^3\right]$$

evaluating the above formula at $a = 1$ m, .1 m, and 1 mm, yields respectively

$$\underline{54.50,\ 54.01,\ \text{and}\ 54.00\ \text{C/m}^3},$$

thus demonstrating the approach to the exact value as Δv gets smaller.

3.21. Calculate the divergence of $\mathbf{D}$ at the point specified if

a) $\mathbf{D} = (1/z^2)\left[10xyz\,\mathbf{a}_x + 5x^2z\,\mathbf{a}_y + (2z^3 - 5x^2y)\,\mathbf{a}_z\right]$ at $P(-2, 3, 5)$: We find

$$\nabla \cdot \mathbf{D} = \left[\frac{10y}{z} + 0 + 2 + \frac{10x^2y}{z^3}\right]_{(-2,3,5)} = \underline{8.96}$$

b) $\mathbf{D} = 5z^2\mathbf{a}_\rho + 10\rho z\,\mathbf{a}_z$ at $P(3, -45°, 5)$: In cylindrical coordinates, we have

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} = \left[\frac{5z^2}{\rho} + 10\rho\right]_{(3,-45°,5)} = \underline{71.67}$$

c) $\mathbf{D} = 2r\sin\theta\sin\phi\,\mathbf{a}_r + r\cos\theta\sin\phi\,\mathbf{a}_\theta + r\cos\phi\,\mathbf{a}_\phi$ at $P(3, 45°, -45°)$: In spherical coordinates, we have

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$
$$= \left[6\sin\theta\sin\phi + \frac{\cos 2\theta\sin\phi}{\sin\theta} - \frac{\sin\phi}{\sin\theta}\right]_{(3,45°,-45°)} = \underline{-2}$$

3.22. Let $\mathbf{D} = 8\rho \sin\phi \, \mathbf{a}_\rho + 4\rho \cos\phi \, \mathbf{a}_\phi$ C/m^2.

a) Find div **D**: Using the divergence formula for cylindrical coordinates (see problem 3.21), we find $\nabla \cdot \mathbf{D} = \underline{12 \sin\phi}$.

b) Find the volume charge density at $P(2.6, 38^\circ, -6.1)$: Since $\rho_v = \nabla \cdot \mathbf{D}$, we evaluate the result of part a at this point to find $\rho_{vP} = 12 \sin 38^\circ = \underline{7.39 \,\text{C/m}^3}$.

c) How much charge is located inside the region defined by $0 < \rho < 1.8$, $20^\circ < \phi < 70^\circ$, $2.4 < z < 3.1$? We use

$$Q = \int_{vol} \rho_v dv = \int_{2.4}^{3.1} \int_{20^\circ}^{70^\circ} \int_0^{1.8} 12 \sin\phi \rho \, d\rho \, d\phi \, dz = -(3.1 - 2.4) 12 \cos\phi \Big|_{20^\circ}^{70^\circ} \frac{\rho^2}{2}\Big|_0^{1.8}$$
$$= \underline{8.13 \text{ C}}$$

3.23. a) A point charge Q lies at the origin. Show that div **D** is zero everywhere except at the origin. For a point charge at the origin we know that $\mathbf{D} = Q/(4\pi r^2) \, \mathbf{a}_r$. Using the formula for divergence in spherical coordinates (see problem 3.21 solution), we find in this case that

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{Q}{4\pi r^2} \right) = 0$$

The above is true provided $r > 0$. When $r = 0$, we have a singularity in **D**, so its divergence is not defined.

b) Replace the point charge with a uniform volume charge density ρ_{v0} for $0 < r < a$. Relate ρ_{v0} to Q and a so that the total charge is the same. Find div **D** everywhere: To achieve the same net charge, we require that $(4/3)\pi a^3 \rho_{v0} = Q$, so $\underline{\rho_{v0} = 3Q/(4\pi a^3) \text{ C/m}^3}$. Gauss' law tells us that inside the charged sphere

$$4\pi r^2 D_r = \frac{4}{3} \pi r^3 \rho_{v0} = \frac{Q r^3}{a^3}$$

Thus

$$D_r = \frac{Qr}{4\pi a^3} \text{ C/m}^2 \text{ and } \nabla \cdot \mathbf{D} = \frac{1}{r^2} \frac{d}{dr} \left(\frac{Q r^3}{4\pi a^3} \right) = \frac{3Q}{4\pi a^3}$$

as expected. Outside the charged sphere, $\mathbf{D} = Q/(4\pi r^2) \, \mathbf{a}_r$ as before, and the divergence is zero.

3.24. Inside the cylindrical shell, $3 < \rho < 4$ m, the electric flux density is given as

$$\mathbf{D} = 5(\rho - 3)^3 \, \mathbf{a}_\rho \text{ C/m}^2$$

a) What is the volume charge density at $\rho = 4$ m? In this case we have

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{\rho} \frac{d}{d\rho} (\rho D_\rho) = \frac{1}{\rho} \frac{d}{d\rho} [5\rho(\rho - 3)^3] = \frac{5(\rho - 3)^2}{\rho} (4\rho - 3) \text{ C/m}^3$$

Evaluating this at $\rho = 4$ m, we find $\rho_v(4) = \underline{16.25 \text{ C/m}^3}$

b) What is the electric flux density at $\rho = 4$ m? We evaluate the given **D** at this point to find $\mathbf{D}(4) = \underline{5 \, \mathbf{a}_\rho \text{ C/m}^2}$

3.24c. How much electric flux leaves the closed surface $3 < \rho < 4$, $0 < \phi < 2\pi$, $-2.5 < z < 2.5$? We note that **D** has only a radial component, and so flux would leave only through the cylinder sides. Also, **D** does not vary with ϕ or z, so the flux is found by a simple product of the side area and the flux density. We further note that $\mathbf{D} = 0$ at $\rho = 3$, so only the outer side (at $\rho = 4$) will contribute. We use the result of part b, and write the flux as

$$\Phi = [2.5 - (-2.5)]2\pi(4)(5) = \underline{200\pi \text{ C}}$$

d) How much charge is contained within the volume used in part c? By Gauss' law, this will be the same as the net outward flux through that volume, or again, $\underline{200\pi \text{ C}}$.

3.25. Within the spherical shell, $3 < r < 4$ m, the electric flux density is given as

$$\mathbf{D} = 5(r-3)^3 \, \mathbf{a}_r \text{ C/m}^2$$

a) What is the volume charge density at $r = 4$? In this case we have

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{5}{r}(r-3)^2(5r-6) \text{ C/m}^3$$

which we evaluate at $r = 4$ to find $\rho_v(r=4) = \underline{17.50 \text{ C/m}^3}$.

b) What is the electric flux density at $r = 4$? Substitute $r = 4$ into the given expression to find $\mathbf{D}(4) = \underline{5\,\mathbf{a}_r \text{ C/m}^2}$

c) How much electric flux leaves the sphere $r = 4$? Using the result of part b, this will be $\Phi = 4\pi(4)^2(5) = \underline{320\pi \text{ C}}$

d) How much charge is contained within the sphere, $r = 4$? From Gauss' law, this will be the same as the outward flux, or again, $Q = \underline{320\pi \text{ C}}$.

3.26. Given the field

$$\mathbf{D} = \frac{5\sin\theta\cos\phi}{r}\,\mathbf{a}_r \text{ C/m}^2,$$

find:

a) the volume charge density: Use

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \underline{\frac{5\sin\theta\cos\phi}{r^2} \text{ C/m}^3}$$

b) the total charge contained within the region $r < 2$ m: To find this, we integrate over the volume:

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \frac{5\sin\theta\cos\phi}{r^2}\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

Before plunging into this one notice that the ϕ integration is of $\cos\phi$ from zero to 2π. This yields a zero result, and so the total enclosed charge is $\underline{Q = 0}$.

c) the value of **D** at the surface $r = 2$: Substituting $r = 2$ into the given field produces

$$\mathbf{D}(r=2) = \underline{\frac{5}{2}\sin\theta\cos\phi\,\mathbf{a}_r \text{ C/m}^2}$$

3.26d. the total electric flux leaving the surface $r = 2$ Since the total enclosed charge is zero (from part *b*), the net outward flux is also zero, from Gauss' law.

3.27. Let $\mathbf{D} = 5.00r^2\mathbf{a}_r$ mC/m^2 for $r \leq 0.08$ m and $\mathbf{D} = 0.205\,\mathbf{a}_r/r^2$ μC/m^2 for $r \geq 0.08$ m (note error in problem statement).

a) Find ρ_v for $r = 0.06$ m: This radius lies within the first region, and so

$$\rho_v = \nabla\cdot\mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{1}{r^2}\frac{d}{dr}(5.00r^4) = 20r \text{ mC/m}^3$$

which when evaluated at $r = 0.06$ yields $\rho_v(r = .06) = \underline{1.20 \text{ mC/m}^3}$.

b) Find ρ_v for $r = 0.1$ m: This is in the region where the second field expression is valid. The $1/r^2$ dependence of this field yields a zero divergence (shown in Problem 3.23), and so the volume charge density is zero at 0.1 m.

c) What surface charge density could be located at $r = 0.08$ m to cause $\mathbf{D} = 0$ for $r > 0.08$ m? The total surface charge should be equal and opposite to the total volume charge. The latter is

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{.08} 20r(\text{mC/m}^3)\,r^2\sin\theta\,dr\,d\theta\,d\phi = 2.57\times10^{-3}\text{ mC} = 2.57\,\mu\text{C}$$

So now

$$\rho_s = -\left[\frac{2.57}{4\pi(.08)^2}\right] = \underline{-32\,\mu\text{C/m}^2}$$

3.28. The electric flux density is given as $\mathbf{D} = 20\rho^3\,\mathbf{a}_\rho$ C/m^2 for $\rho < 100\,\mu$m, and $k\,\mathbf{a}_\rho/\rho$ for $\rho > 100\,\mu$m.

a) Find k so that $\mathbf{D}$ is continuous at $\rho = 100\,\mu$m: We require

$$20\times10^{-12} = \frac{k}{10^{-4}} \quad\Rightarrow\quad k = \underline{2\times10^{-15}\text{ C/m}}$$

b) Find and sketch ρ_v as a function of ρ: In cylindrical coordinates, with only a radial component of $\mathbf{D}$, we use

$$\rho_v = \nabla\cdot\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(20\rho^4) = \underline{80\rho^2\text{ C/m}^3\ \ (\rho < 100\,\mu\text{m})}$$

For $\rho > 100\,\mu$m, we obtain

$$\rho_v = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{k}{\rho}) = \underline{0}$$

The sketch of ρ_v vs. ρ would be a parabola, starting at the origin, reaching a maximum value of 8×10^{-7} C/m^3 at $\rho = 100\,\mu$m. The plot is zero at larger radii.

3.29. In the region of free space that includes the volume $2 < x, y, z < 3$,

$$\mathbf{D} = \frac{2}{z^2}(yz\,\mathbf{a}_x + xz\,\mathbf{a}_y - 2xy\,\mathbf{a}_z)\text{ C/m}^2$$

a) Evaluate the volume integral side of the divergence theorem for the volume defined above: In cartesian, we find $\nabla\cdot\mathbf{D} = 8xy/z^3$. The volume integral side is now

$$\int_{vol}\nabla\cdot\mathbf{D}\,dv = \int_2^3\int_2^3\int_2^3\frac{8xy}{z^3}\,dxdydz = (9-4)(9-4)\left(\frac{1}{4}-\frac{1}{9}\right) = \underline{3.47\text{ C}}$$

3.29b. Evaluate the surface integral side for the corresponding closed surface: We call the surfaces at $x = 3$ and $x = 2$ the front and back surfaces respectively, those at $y = 3$ and $y = 2$ the right and left surfaces, and those at $z = 3$ and $z = 2$ the top and bottom surfaces. To evaluate the surface integral side, we integrate $\mathbf{D} \cdot \mathbf{n}$ over all six surfaces and sum the results. Note that since the x component of $\mathbf{D}$ does not vary with x, the outward fluxes from the front and back surfaces will cancel each other. The same is true for the left and right surfaces, since D_y does not vary with y. This leaves only the top and bottom surfaces, where the fluxes are:

$$\oint \mathbf{D} \cdot d\mathbf{S} = \underbrace{\int_2^3 \int_2^3 \frac{-4xy}{3^2}\, dx dy}_{\text{top}} - \underbrace{\int_2^3 \int_2^3 \frac{-4xy}{2^2}\, dx dy}_{\text{bottom}} = (9-4)(9-4)\left(\frac{1}{4} - \frac{1}{9}\right) = \underline{3.47 \text{ C}}$$

3.30. If $\mathbf{D} = 15\rho^2 \sin 2\phi\, \mathbf{a}_\rho + 10\rho^2 \cos 2\phi\, \mathbf{a}_\phi$ C/m^2, evaluate both sides of the divergence theorem for the region $1 < \rho < 2$ m, $1 < \phi < 2$ rad, $1 < z < 2$ m: Taking the surface integral side first, the six sides over which the flux must be evaluated are only four, since there is no z component of $\mathbf{D}$. We are left with the sides at $\phi = 1$ and $\phi = 2$ rad (left and right sides, respectively), and those at $\rho = 1$ and $\rho = 2$ (back and front sides). We evaluate

$$\oint \mathbf{D} \cdot d\mathbf{S} = \underbrace{\int_1^2 \int_1^2 15(2)^2 \sin(2\phi)\,(2) d\phi dz}_{\text{front}} - \underbrace{\int_1^2 \int_1^2 15(1)^2 \sin(2\phi)\,(1) d\phi dz}_{\text{back}}$$

$$- \underbrace{\int_1^2 \int_1^2 10\rho^2 \cos(2)\, d\rho dz}_{\text{left}} + \underbrace{\int_1^2 \int_1^2 10\rho^2 \cos(4)\, d\rho dz}_{\text{right}} = \underline{6.93 \text{ C}}$$

For the volume integral side, we first evaluate the divergence of $\mathbf{D}$, which is

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(15\rho^3 \sin 2\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(10\rho^2 \cos 2\phi) = 25\rho \sin 2\phi$$

Next

$$\int_{vol} \nabla \cdot \mathbf{D}\, dv = \int_1^2 \int_1^2 \int_1^2 25\rho \sin(2\phi)\, \rho d\rho\, d\phi\, dz = \frac{25}{3}\rho^3\Big|_1^2 \left[\frac{-\cos(2\phi)}{2}\right]_1^2 = \underline{6.93 \text{ C}}$$

3.31. Given the flux density

$$\mathbf{D} = \frac{16}{r}\cos(2\theta)\, \mathbf{a}_\theta \text{ C/m}^2,$$

use two different methods to find the total charge within the region $1 < r < 2$ m, $1 < \theta < 2$ rad, $1 < \phi < 2$ rad: We use the divergence theorem and first evaluate the surface integral side. We are evaluating the net outward flux through a curvilinear "cube", whose boundaries are defined by the specified ranges. The flux contributions will be only through the surfaces of constant θ, however, since $\mathbf{D}$ has only a θ component. On a constant-theta surface, the differential area is $da = r \sin\theta dr d\phi$, where θ is fixed at the surface location. Our flux integral becomes

$$\oint \mathbf{D} \cdot d\mathbf{S} = -\underbrace{\int_1^2 \int_1^2 \frac{16}{r}\cos(2)\, r \sin(1)\, dr d\phi}_{\theta=1} + \underbrace{\int_1^2 \int_1^2 \frac{16}{r}\cos(4)\, r \sin(2)\, dr d\phi}_{\theta=2}$$

$$= -16\left[\cos(2)\sin(1) - \cos(4)\sin(2)\right] = \underline{-3.91 \text{ C}}$$

3.31. (continued) We next evaluate the volume integral side of the divergence theorem, where in this case,

$$\nabla \cdot \mathbf{D} = \frac{1}{r \sin\theta} \frac{d}{d\theta}(\sin\theta \, D_\theta) = \frac{1}{r \sin\theta} \frac{d}{d\theta}\left[\frac{16}{r} \cos 2\theta \sin\theta\right] = \frac{16}{r^2}\left[\frac{\cos 2\theta \cos\theta}{\sin\theta} - 2 \sin 2\theta\right]$$

We now evaluate:

$$\int_{vol} \nabla \cdot \mathbf{D} \, dv = \int_1^2 \int_1^2 \int_1^2 \frac{16}{r^2}\left[\frac{\cos 2\theta \cos\theta}{\sin\theta} - 2 \sin 2\theta\right] r^2 \sin\theta \, dr d\theta d\phi$$

The integral simplifies to

$$\int_1^2 \int_1^2 \int_1^2 16[\cos 2\theta \cos\theta - 2 \sin 2\theta \sin\theta] \, dr d\theta d\phi = 8 \int_1^2 [3 \cos 3\theta - \cos\theta] \, d\theta = \underline{-3.91 \text{ C}}$$

3.32. If $\mathbf{D} = 2r \, \mathbf{a}_r$ C/m^2, find the total electric flux leaving the surface of the cube, $0 < x, y, z < 0.4$: This is where the divergence theorem really saves you time! First find

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2} \frac{d}{dr}(r^2 \times 2r) = 6$$

Then the net outward flux will be

$$\int_{vol} \nabla \cdot \mathbf{D} \, dv = 6(0.4)^3 = \underline{0.38 \text{ C}}$$