CHAPTER 4

4.1. The value of $\mathbf{E}$ at $P(\rho = 2,\ \phi = 40^\circ,\ z = 3)$ is given as $\mathbf{E} = 100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z$ V/m. Determine the incremental work required to move a 20 μC charge a distance of 6 μm:

a) in the direction of $\mathbf{a}_\rho$: The incremental work is given by $dW = -q\,\mathbf{E} \cdot d\mathbf{L}$, where in this case, $d\mathbf{L} = d\rho\,\mathbf{a}_\rho = 6 \times 10^{-6}\,\mathbf{a}_\rho$. Thus

$$dW = -(20 \times 10^{-6}\,\mathrm{C})(100\,\mathrm{V/m})(6 \times 10^{-6}\,\mathrm{m}) = -12 \times 10^{-9}\,\mathrm{J} = \underline{-12\,\mathrm{nJ}}$$

b) in the direction of $\mathbf{a}_\phi$: In this case $d\mathbf{L} = 2\,d\phi\,\mathbf{a}_\phi = 6 \times 10^{-6}\,\mathbf{a}_\phi$, and so

$$dW = -(20 \times 10^{-6})(-200)(6 \times 10^{-6}) = 2.4 \times 10^{-8}\,\mathrm{J} = \underline{24\ \mathrm{nJ}}$$

c) in the direction of $\mathbf{a}_z$: Here, $d\mathbf{L} = dz\,\mathbf{a}_z = 6 \times 10^{-6}\,\mathbf{a}_z$, and so

$$dW = -(20 \times 10^{-6})(300)(6 \times 10^{-6}) = -3.6 \times 10^{-8}\,\mathrm{J} = \underline{-36\ \mathrm{nJ}}$$

d) in the direction of $\mathbf{E}$: Here, $d\mathbf{L} = 6 \times 10^{-6}\,\mathbf{a}_E$, where

$$\mathbf{a}_E = \frac{100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z}{[100^2 + 200^2 + 300^2]^{1/2}} = 0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z$$

Thus

$$\begin{aligned} dW &= -(20 \times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z] \cdot [0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z](6 \times 10^{-6}) \\ &= \underline{-44.9\,\mathrm{nJ}} \end{aligned}$$

e) In the direction of $\mathbf{G} = 2\,\mathbf{a}_x - 3\,\mathbf{a}_y + 4\,\mathbf{a}_z$: In this case, $d\mathbf{L} = 6 \times 10^{-6}\,\mathbf{a}_G$, where

$$\mathbf{a}_G = \frac{2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z}{[2^2 + 3^2 + 4^2]^{1/2}} = 0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z$$

So now

$$\begin{aligned} dW &= -(20 \times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z] \cdot [0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z](6 \times 10^{-6}) \\ &= -(20 \times 10^{-6})\big[37.1(\mathbf{a}_\rho \cdot \mathbf{a}_x) - 55.7(\mathbf{a}_\rho \cdot \mathbf{a}_y) - 74.2(\mathbf{a}_\phi \cdot \mathbf{a}_x) + 111.4(\mathbf{a}_\phi \cdot \mathbf{a}_y) \\ &\quad + 222.9\big]\,(6 \times 10^{-6}) \end{aligned}$$

where, at P, $(\mathbf{a}_\rho \cdot \mathbf{a}_x) = (\mathbf{a}_\phi \cdot \mathbf{a}_y) = \cos(40^\circ) = 0.766$, $(\mathbf{a}_\rho \cdot \mathbf{a}_y) = \sin(40^\circ) = 0.643$, and $(\mathbf{a}_\phi \cdot \mathbf{a}_x) = -\sin(40^\circ) = -0.643$. Substituting these results in

$$dW = -(20 \times 10^{-6})[28.4 - 35.8 + 47.7 + 85.3 + 222.9](6 \times 10^{-6}) = \underline{-41.8\,\mathrm{nJ}}$$

4.2. Let $\mathbf{E} = 400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z$ in the neighborhood of point $P(6, 2, -3)$. Find the incremental work done in moving a 4-C charge a distance of 1 mm in the direction specified by:

a) $\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$: We write

$$dW = -q\mathbf{E}\cdot d\mathbf{L} = -4(400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z)\cdot\frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}}(10^{-3})$$
$$= -\frac{(4\times 10^{-3})}{\sqrt{3}}(400 - 300 + 500) = \underline{-1.39\,\text{J}}$$

b) $-2\mathbf{a}_x + 3\mathbf{a}_y - \mathbf{a}_z$: The computation is similar to that of part a, but we change the direction:

$$dW = -q\mathbf{E}\cdot d\mathbf{L} = -4(400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z)\cdot\frac{(-2\mathbf{a}_x + 3\mathbf{a}_y - \mathbf{a}_z)}{\sqrt{14}}(10^{-3})$$
$$= -\frac{(4\times 10^{-3})}{\sqrt{14}}(-800 - 900 - 500) = \underline{2.35\,\text{J}}$$

4.3. If $\mathbf{E} = 120\,\mathbf{a}_\rho$ V/m, find the incremental amount of work done in moving a 50 μm charge a distance of 2 mm from:

a) $P(1, 2, 3)$ toward $Q(2, 1, 4)$: The vector along this direction will be $Q - P = (1, -1, 1)$ from which $\mathbf{a}_{PQ} = [\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z]/\sqrt{3}$. We now write

$$dW = -q\mathbf{E}\cdot d\mathbf{L} = -(50\times 10^{-6})\left[120\mathbf{a}_\rho\cdot\frac{(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z}{\sqrt{3}}\right](2\times 10^{-3})$$
$$= -(50\times 10^{-6})(120)\left[(\mathbf{a}_\rho\cdot\mathbf{a}_x) - (\mathbf{a}_\rho\cdot\mathbf{a}_y)\right]\frac{1}{\sqrt{3}}(2\times 10^{-3})$$

At P, $\phi = \tan^{-1}(2/1) = 63.4^\circ$. Thus $(\mathbf{a}_\rho\cdot\mathbf{a}_x) = \cos(63.4) = 0.447$ and $(\mathbf{a}_\rho\cdot\mathbf{a}_y) = \sin(63.4) = 0.894$. Substituting these, we obtain $dW = \underline{3.1\,\mu\text{J}}$.

b) $Q(2, 1, 4)$ toward $P(1, 2, 3)$: A little thought is in order here: Note that the field has only a radial component and does not depend on ϕ or z. Note also that P and Q are at the same radius ($\sqrt{5}$) from the z axis, but have different ϕ and z coordinates. We could just as well position the two points at the same z location and the problem would not change. If this were so, then moving along a straight line between P and Q would thus involve moving along a chord of a circle whose radius is $\sqrt{5}$. Halfway along this line is a point of symmetry in the field (make a sketch to see this). This means that when starting from either point, the initial force will be the same. Thus the answer is $dW = \underline{3.1\,\mu\text{J}}$ as in part a. This is also found by going through the same procedure as in part a, but with the direction (roles of P and Q) reversed.

4.4. Find the amount of energy required to move a 6-C charge from the origin to $P(3, 1, -1)$ in the field $\mathbf{E} = 2x\mathbf{a}_x - 3y^2\mathbf{a}_y + 4\mathbf{a}_z$ V/m along the straight-line path $x = -3z$, $y = x + 2z$: We set up the computation as follows, and find the the result *does not depend on the path*.

$$W = -q\int\mathbf{E}\cdot d\mathbf{L} = -6\int(2x\mathbf{a}_x - 3y^2\mathbf{a}_y + 4\mathbf{a}_z)\cdot(dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z)$$
$$= -6\int_0^3 2x\,dx \;+\; 6\int_0^1 3y^2\,dy \;-\; 6\int_0^{-1} 4\,dz = \underline{-24\,\text{J}}$$

4.5. Compute the value of $\int_A^P \mathbf{G}\cdot d\mathbf{L}$ for $\mathbf{G} = 2y\mathbf{a}_x$ with $A(1,-1,2)$ and $P(2,1,2)$ using the path:

a) straight-line segments $A(1,-1,2)$ to $B(1,1,2)$ to $P(2,1,2)$: In general we would have

$$\int_A^P \mathbf{G}\cdot d\mathbf{L} = \int_A^P 2y\,dx$$

The change in x occurs when moving between B and P, during which $y = 1$. Thus

$$\int_A^P \mathbf{G}\cdot d\mathbf{L} = \int_B^P 2y\,dx = \int_1^2 2(1)dx = \underline{2}$$

b) straight-line segments $A(1,-1,2)$ to $C(2,-1,2)$ to $P(2,1,2)$: In this case the change in x occurs when moving from A to C, during which $y = -1$. Thus

$$\int_A^P \mathbf{G}\cdot d\mathbf{L} = \int_A^C 2y\,dx = \int_1^2 2(-1)dx = \underline{-2}$$

4.6. Let $\mathbf{G} = 4x\mathbf{a}_x + 2z\mathbf{a}_y + 2y\mathbf{a}_z$. Given an initial point $P(2,1,1)$ and a final point $Q(4,3,1)$, find $\int \mathbf{G}\cdot d\mathbf{L}$ using the path: a) straight line: $y = x - 1$, $z = 1$; b) parabola: $6y = x^2 + 2$, $z = 1$:

With $\mathbf{G}$ as given, the line integral will be

$$\int \mathbf{G}\cdot d\mathbf{L} = \int_2^4 4x\,dx + \int_1^3 2z\,dy + \int_1^1 2y\,dz$$

Clearly, we are going nowhere in z, so the last integral is zero. With $z = 1$, the first two evaluate as

$$\int \mathbf{G}\cdot d\mathbf{L} = 2x^2\Big|_2^4 + 2y\Big|_1^3 = \underline{28}$$

The paths specified in parts a and b did not play a role, meaning that the integral between the specified points is path-independent.

4.7. Repeat Problem 4.6 for $\mathbf{G} = 3xy^3\mathbf{a}_x + 2z\mathbf{a}_y$. Now things are different in that the path does matter:

a) straight line: $y = x - 1$, $z = 1$: We obtain:

$$\int \mathbf{G}\cdot d\mathbf{L} = \int_2^4 3xy^2\,dx + \int_1^3 2z\,dy = \int_2^4 3x(x-1)^2\,dx + \int_1^3 2(1)\,dy = \underline{90}$$

b) parabola: $6y = x^2 + 2$, $z = 1$: We obtain:

$$\int \mathbf{G}\cdot d\mathbf{L} = \int_2^4 3xy^2\,dx + \int_1^3 2z\,dy = \int_2^4 \frac{1}{12}x(x^2+2)^2\,dx + \int_1^3 2(1)\,dy = \underline{82}$$

4.8. A point charge Q_1 is located at the origin in free space. Find the work done in carrying a charge Q_2 from: (a) $B(r_B, \theta_B, \phi_B)$ to $C(r_A, \theta_B, \phi_B)$ with θ and ϕ held constant; (b) $C(r_A, \theta_B, \phi_B)$ to $D(r_A, \theta_A, \phi_B)$ with r and ϕ held constant; (c) $D(r_A, \theta_A, \phi_B)$ to $A(r_A, \theta_A, \phi_A)$ with r and θ held constant: The general expression for the work done in this instance is

$$W = -Q_2 \int \mathbf{E} \cdot d\mathbf{L} = -Q_2 \int \frac{Q_1}{4\pi\epsilon_0 r^2} \mathbf{a}_r \cdot (dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi) = -\frac{Q_1 Q_2}{4\pi\epsilon_0} \int \frac{dr}{r^2}$$

We see that only changes in r will produce non-zero results. Thus for part a we have

$$W = -\frac{Q_1 Q_2}{4\pi\epsilon_0} \int_{r_B}^{r_A} \frac{dr}{r^2} = \underline{\frac{Q_1 Q_2}{4\pi\epsilon_0} \left[\frac{1}{r_A} - \frac{1}{r_B} \right] \text{ J}}$$

The answers to parts b and c (involving paths over which r is held constant) are both $\underline{0}$.

4.9. A uniform surface charge density of 20 nC/m^2 is present on the spherical surface $r = 0.6\,\text{cm}$ in free space.

a) Find the absolute potential at $P(r = 1\,\text{cm}, \theta = 25^\circ, \phi = 50^\circ)$: Since the charge density is uniform and is spherically-symmetric, the angular coordinates do not matter. The potential function for $r > 0.6$ cm will be that of a point charge of $Q = 4\pi a^2 \rho_s$, or

$$V(r) = \frac{4\pi(0.6 \times 10^{-2})^2 (20 \times 10^{-9})}{4\pi\epsilon_0 r} = \frac{0.081}{r} \text{ V} \quad \text{with } r \text{ in meters}$$

At $r = 1\,\text{cm}$, this becomes $V(r = 1\,\text{cm}) = \underline{8.14 \text{ V}}$

b) Find V_{AB} given points $A(r = 2\,\text{cm}, \theta = 30^\circ, \phi = 60^\circ)$ and $B(r = 3\,\text{cm}, \theta = 45^\circ, \phi = 90^\circ)$: Again, the angles do not matter because of the spherical symmetry. We use the part a result to obtain

$$V_{AB} = V_A - V_B = 0.081 \left[\frac{1}{0.02} - \frac{1}{0.03} \right] = \underline{1.36 \text{ V}}$$

4.10. Given a surface charge density of 8 nC/m^2 on the plane $x = 2$, a line charge density of 30 nC/m on the line $x = 1$, $y = 2$, and a 1-μC point charge at $P(-1, -1, 2)$, find V_{AB} for points $A(3, 4, 0)$ and $B(4, 0, 1)$: We need to find a potential function for the combined charges. That for the point charge we know to be

$$V_p(r) = \frac{Q}{4\pi\epsilon_0 r}$$

Potential functions for the sheet and line charges can be found by taking indefinite integrals of the electric fields for those distributions. For the line charge, we have

$$V_l(\rho) = -\int \frac{\rho_l}{2\pi\epsilon_0 \rho} d\rho + C_1 = -\frac{\rho_l}{2\pi\epsilon_0} \ln(\rho) + C_1$$

For the sheet charge, we have

$$V_s(x) = -\int \frac{\rho_s}{2\epsilon_0} dx + C_2 = -\frac{\rho_s}{2\epsilon_0} x + C_2$$

4.10. (continued) The total potential function will be the sum of the three. Combining the integration constants, we obtain:

$$V = \frac{Q}{4\pi\epsilon_0 r} - \frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) - \frac{\rho_s}{2\epsilon_0}x + C$$

The terms in this expression are not referenced to a common origin, since the charges are at different positions. The parameters r, ρ, and x are *scalar distances* from the charges, and will be treated as such here. For point A we have $r_A = \sqrt{(3-(-1))^2 + (4-(-1))^2 + (-2)^2} = \sqrt{45}$, $\rho_A = \sqrt{(3-1)^2 + (4-2)^2} = \sqrt{8}$, and its distance from the sheet charge is $x_A = 3 - 2 = 1$. The potential at A is then

$$V_A = \frac{10^{-6}}{4\pi\epsilon_0\sqrt{45}} - \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\sqrt{8} - \frac{8\times10^{-9}}{2\epsilon_0}(1) + C$$

At point B, $r_B = \sqrt{(4-(-1))^2 + (0-(-1))^2 + (1-2)^2} = \sqrt{27}$,
$\rho_B = \sqrt{(4-1)^2 + (0-2)^2} = \sqrt{13}$, and the distance from the sheet charge is $x_B = 4 - 2 = 2$.
The potential at A is then

$$V_B = \frac{10^{-6}}{4\pi\epsilon_0\sqrt{27}} - \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\sqrt{13} - \frac{8\times10^{-9}}{2\epsilon_0}(2) + C$$

Then

$$V_A - V_B = \frac{10^{-6}}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{45}} - \frac{1}{\sqrt{27}}\right] - \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\left(\sqrt{\frac{8}{13}}\right) - \frac{8\times10^{-9}}{2\epsilon_0}(1-2) = \underline{193\text{ V}}$$

4.11. Let a uniform surface charge density of $5\,\text{nC/m}^2$ be present at the $z = 0$ plane, a uniform line charge density of $8\,\text{nC/m}$ be located at $x = 0$, $z = 4$, and a point charge of $2\,\mu\text{C}$ be present at $P(2, 0, 0)$. If $V = 0$ at $M(0, 0, 5)$, find V at $N(1, 2, 3)$: We need to find a potential function for the combined charges which is zero at M. That for the point charge we know to be

$$V_p(r) = \frac{Q}{4\pi\epsilon_0 r}$$

Potential functions for the sheet and line charges can be found by taking indefinite integrals of the electric fields for those distributions. For the line charge, we have

$$V_l(\rho) = -\int \frac{\rho_l}{2\pi\epsilon_0\rho}d\rho + C_1 = -\frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) + C_1$$

For the sheet charge, we have

$$V_s(z) = -\int \frac{\rho_s}{2\epsilon_0}dz + C_2 = -\frac{\rho_s}{2\epsilon_0}z + C_2$$

The total potential function will be the sum of the three. Combining the integration constants, we obtain:

$$V = \frac{Q}{4\pi\epsilon_0 r} - \frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) - \frac{\rho_s}{2\epsilon_0}z + C$$

4.11. (continued) The terms in this expression are not referenced to a common origin, since the charges are at different positions. The parameters r, ρ, and z are *scalar distances* from the charges, and will be treated as such here. To evaluate the constant, C, we first look at point M, where $V_T = 0$. At M, $r = \sqrt{2^2 + 5^2} = \sqrt{29}$, $\rho = 1$, and $z = 5$. We thus have

$$0 = \frac{2 \times 10^{-6}}{4\pi\epsilon_0\sqrt{29}} - \frac{8 \times 10^{-9}}{2\pi\epsilon_0}\ln(1) - \frac{5 \times 10^{-9}}{2\epsilon_0}5 + C \quad \Rightarrow \quad C = -1.93 \times 10^3 \text{ V}$$

At point N, $r = \sqrt{1+4+9} = \sqrt{14}$, $\rho = \sqrt{2}$, and $z = 3$. The potential at N is thus

$$V_N = \frac{2 \times 10^{-6}}{4\pi\epsilon_0\sqrt{14}} - \frac{8 \times 10^{-9}}{2\pi\epsilon_0}\ln(\sqrt{2}) - \frac{5 \times 10^{-9}}{2\epsilon_0}(3) - 1.93 \times 10^3 = 1.98 \times 10^3 \text{ V} = \underline{1.98\,\text{kV}}$$

4.12. Three point charges, $0.4\,\mu$C each, are located at $(0, 0, -1)$, $(0, 0, 0)$, and $(0, 0, 1)$, in free space.

a) Find an expression for the absolute potential as a function of z along the line $x = 0$, $y = 1$: From a point located at position z along the given line, the distances to the three charges are $R_1 = \sqrt{(z-1)^2 + 1}$, $R_2 = \sqrt{z^2 + 1}$, and $R_3 = \sqrt{(z+1)^2 + 1}$. The total potential will be

$$V(z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right]$$

Using $q = 4 \times 10^{-7}$ C, this becomes

$$\underline{V(z) = (3.6 \times 10^3)\left[\frac{1}{\sqrt{(z-1)^2+1}} + \frac{1}{\sqrt{z^2+1}} + \frac{1}{\sqrt{(z+1)^2+1}}\right] \text{ V}}$$

b) Sketch $V(z)$. The sketch will show that V maximizes to a value of 8.68×10^3 at $z = 0$, and then monotonically decreases with increasing $|z|$ symmetrically on either side of $z = 0$.

4.13. Three identical point charges of 4 pC each are located at the corners of an equilateral triangle 0.5 mm on a side in free space. How much work must be done to move one charge to a point equidistant from the other two and on the line joining them? This will be the magnitude of the charge times the potential difference between the finishing and starting positions, or

$$W = \frac{(4 \times 10^{-12})^2}{2\pi\epsilon_0}\left[\frac{1}{2.5} - \frac{1}{5}\right] \times 10^4 = 5.76 \times 10^{-10} \text{ J} = \underline{576\,\text{pJ}}$$

4.14. two 6-nC point charges are located at $(1, 0, 0)$ and $(-1, 0, 0)$ in free space.

a) Find V at $P(0, 0, z)$: Since the charges are positioned symmetrically about the z axis, the potential at z will be double that from one charge. This becomes:

$$V(z) = (2)\frac{q}{4\pi\epsilon_0\sqrt{z^2+1}} = \underline{\frac{q}{2\pi\epsilon_0\sqrt{z^2+1}}}$$

b) Find V_{max}: It is clear from the part a result that V will maximize at $z = 0$, or $v_{max} = q/(2\pi\epsilon_0) = \underline{108\text{ V}}$.

4.14. (continued)

c) Calculate $|dV/dz|$ on the z axis: Differentiating the part a result, we find

$$\left|\frac{dV}{dz}\right| = \underline{\frac{qz}{\pi\epsilon_0(z^2+1)^{3/2}}\ \text{V/m}}$$

d) Find $|dV/dz|_{max}$: To find this we need to differentiate the part c result and find its zero:

$$\frac{d}{dz}\left|\frac{dV}{dz}\right| = \frac{q(1-2z^2)}{\pi\epsilon_0(z^2+1)^{5/2}} = 0 \quad \Rightarrow \quad z = \pm\frac{1}{\sqrt{2}}$$

Substituting $z = 1/\sqrt{2}$ into the part c result, we find

$$\left|\frac{dV}{dz}\right|_{max} = \frac{q}{\sqrt{2}\pi\epsilon_0(3/2)^{3/2}} = \underline{83.1\ \text{V/m}}$$

4.15. Two uniform line charges, 8 nC/m each, are located at $x = 1$, $z = 2$, and at $x = -1$, $y = 2$ in free space. If the potential at the origin is 100 V, find V at $P(4, 1, 3)$: The net potential function for the two charges would in general be:

$$V = -\frac{\rho_l}{2\pi\epsilon_0}\ln(R_1) - \frac{\rho_l}{2\pi\epsilon_0}\ln(R_2) + C$$

At the origin, $R_1 = R_2 = \sqrt{5}$, and $V = 100$ V. Thus, with $\rho_l = 8 \times 10^{-9}$,

$$100 = -2\frac{(8\times10^{-9})}{2\pi\epsilon_0}\ln(\sqrt{5}) + C \quad \Rightarrow \quad C = 331.6\ \text{V}$$

At $P(4, 1, 3)$, $R_1 = |(4, 1, 3) - (1, 1, 2)| = \sqrt{10}$ and $R_2 = |(4, 1, 3) - (-1, 2, 3)| = \sqrt{26}$. Therefore

$$V_P = -\frac{(8\times10^{-9})}{2\pi\epsilon_0}\left[\ln(\sqrt{10}) + \ln(\sqrt{26})\right] + 331.6 = \underline{-68.4\ \text{V}}$$

4.16. Uniform surface charge densities of 6, 4, and 2 nC/m^2 are present at $r = 2$, 4, and 6 cm, respectively, in free space.

a) Assume $V = 0$ at infinity, and find $V(r)$. We keep in mind the definition of absolute potential as the work done in moving a unit positive charge from infinity to location r. At radii outside all three spheres, the potential will be the same as that of a point charge at the origin, whose charge is the sum of the three sphere charges:

$$V(r)\,(r > 6\,\text{cm}) = \frac{q_1 + q_2 + q_3}{4\pi\epsilon_0 r} = \frac{[4\pi(.02)^2(6) + 4\pi(.04)^2(4) + 4\pi(.06)^2(2)] \times 10^{-9}}{4\pi\epsilon_0 r}$$
$$= \frac{(96 + 256 + 288)\pi \times 10^{-13}}{4\pi(8.85 \times 10^{-12})r} = \underline{\frac{1.81}{r}\text{ V}} \text{ where r is in meters}$$

As the unit charge is moved inside the outer sphere to positions $4 < r < 6$ cm, the outer sphere contribution to the energy is fixed at its value at $r = 6$. Therefore,

$$V(r)\,(4 < r < 6\,\text{cm}) = \frac{q_1 + q_2}{4\pi\epsilon_0 r} + \frac{q_3}{4\pi\epsilon_0(.06)} = \underline{\frac{0.994}{r} + 13.6\text{ V}}$$

In moving inside the sphere at $r = 4$ cm, the contribution from that sphere becomes fixed at its potential function at $r = 4$:

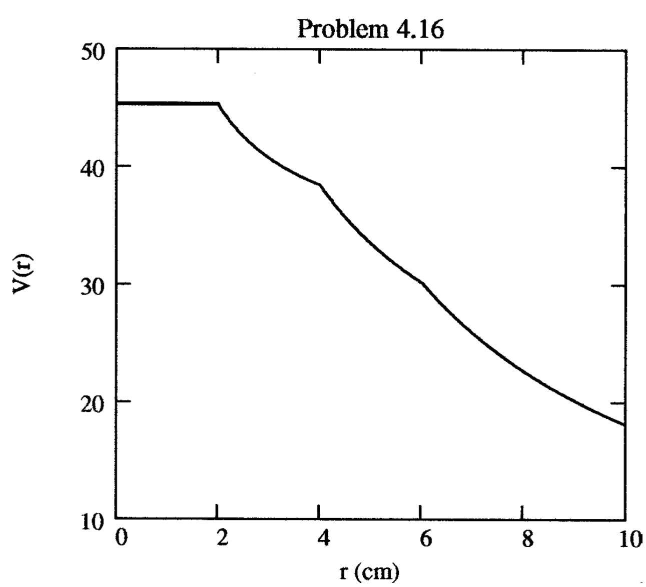
$$V(r)\,(2 < r < 4\,\text{cm}) = \frac{q_1}{4\pi\epsilon_0 r} + \frac{q_2}{4\pi\epsilon_0(.04)} + \frac{q_3}{4\pi\epsilon_0(.06)} = \underline{\frac{0.271}{r} + 31.7\text{ V}}$$

Finally, using the same reasoning, the potential inside the inner sphere becomes

$$V(r)\,(r < 2\,\text{cm}) = \frac{0.271}{.02} + 31.7 = \underline{45.3\text{ V}}$$

b) Calculate V at $r = 1, 3, 5,$ and 7 cm: Using the results of part a, we substitute these distances (in meters) into the appropriate formulas to obtain: $V(1) = \underline{45.3\text{ V}}$, $V(3) = \underline{40.7\text{ V}}$, $V(5) = \underline{33.5\text{ V}}$, and $V(7) = \underline{25.9\text{ V}}$.

c) Sketch V versus r for $0 < r < 10$ cm.



4.17. Uniform surface charge densities of 6 and 2 nC/m^2 are present at $\rho = 2$ and 6 cm respectively, in free space. Assume $V = 0$ at $\rho = 4$ cm, and calculate V at:

a) $\rho = 5$ cm: Since $V = 0$ at 4 cm, the potential at 5 cm will be the potential difference between points 5 and 4:

$$V_5 = -\int_4^5 \mathbf{E}\cdot d\mathbf{L} = -\int_4^5 \frac{a\rho_{sa}}{\epsilon_0 \rho}\, d\rho = -\frac{(.02)(6\times 10^{-9})}{\epsilon_0}\ln\left(\frac{5}{4}\right) = \underline{-3.026\,\text{V}}$$

b) $\rho = 7$ cm: Here we integrate piecewise from $\rho = 4$ to $\rho = 7$:

$$V_7 = -\int_4^6 \frac{a\rho_{sa}}{\epsilon_0\rho} d\rho - \int_6^7 \frac{(a\rho_{sa} + b\rho_{sb})}{\epsilon_0 \rho} d\rho$$

With the given values, this becomes

$$V_7 = -\left[\frac{(.02)(6\times 10^{-9})}{\epsilon_0}\right]\ln\left(\frac{6}{4}\right) - \left[\frac{(.02)(6\times 10^{-9}) + (.06)(2\times 10^{-9})}{\epsilon_0}\right]\ln\left(\frac{7}{6}\right)$$
$$= \underline{-9.678\,\text{V}}$$

4.18. A nonuniform linear charge density, $\rho_L = 8/(z^2+1)$ nC/m lies along the z axis. Find the potential at $P(\rho = 1, 0, 0)$ in free space if $V = 0$ at infinity: This last condition enables us to write the potential at P as a superposition of point charge potentials. The result is the integral:

$$V_P = \int_{-\infty}^{\infty} \frac{\rho_L dz}{4\pi\epsilon_0 R}$$

where $R = \sqrt{z^2+1}$ is the distance from a point z on the z axis to P. Substituting the given charge distribution and R into the integral gives us

$$V_P = \int_{-\infty}^{\infty} \frac{8\times 10^{-9} dz}{4\pi\epsilon_0 (z^2+1)^{3/2}} = \frac{2\times 10^{-9}}{\pi\epsilon_0} \frac{z}{\sqrt{z^2+1}}\Big|_{-\infty}^{\infty} = \underline{144\text{ V}}$$

4.19. The annular surface, $1\,\text{cm} < \rho < 3\,\text{cm}$, $z = 0$, carries the nonuniform surface charge density $\rho_s = 5\rho\,\text{nC/m}^2$. Find V at $P(0, 0, 2\,\text{cm})$ if $V = 0$ at infinity: We use the superposition integral form:

$$V_P = \int\!\!\int \frac{\rho_s\, da}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|}$$

where $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = \rho\mathbf{a}_\rho$. We integrate over the surface of the annular region, with $da = \rho\, d\rho\, d\phi$. Substituting the given values, we find

$$V_P = \int_0^{2\pi}\int_{.01}^{.03} \frac{(5\times 10^{-9})\rho^2\, d\rho\, d\phi}{4\pi\epsilon_0\sqrt{\rho^2 + z^2}}$$

Substituting $z = .02$, and using tables, the integral evaluates as

$$V_P = \left[\frac{(5\times 10^{-9})}{2\epsilon_0}\right]\left[\frac{\rho}{2}\sqrt{\rho^2 + (.02)^2} - \frac{(.02)^2}{2}\ln(\rho + \sqrt{\rho^2 + (.02)^2})\right]_{.01}^{.03} = \underline{.081\text{ V}}$$

4.20. Fig. 4.11 shows three separate charge distributions in the $z = 0$ plane in free space.

a) find the total charge for each distribution: Line charge along the y axis:

$$Q_1 = \int_3^5 \pi \times 10^{-9} dy = 2\pi \times 10^{-9}\,\text{C} = \underline{6.28\,\text{nC}}$$

Line charge in an arc at radius $\rho = 3$:

$$Q_2 = \int_{10^\circ}^{70^\circ} (10^{-9})\,3\,d\phi = 4.5 \times 10^{-9}\,(70 - 10)\frac{2\pi}{360} = 4.71 \times 10^{-9}\,\text{C} = \underline{4.71\,\text{nC}}$$

Sheet charge:

$$Q_3 = \int_{10^\circ}^{70^\circ} \int_{1.6}^{3.5} (10^{-9})\,\rho\,d\rho\,d\phi = 5.07 \times 10^{-9}\,\text{C} = \underline{5.07\,\text{nC}}$$

b) Find the potential at $P(0, 0, 6)$ caused by each of the three charge distributions acting alone: Line charge along y axis:

$$V_{P1} = \int_3^5 \frac{\rho_L dL}{4\pi\epsilon_0 R} = \int_3^5 \frac{\pi \times 10^{-9} dy}{4\pi\epsilon_0\sqrt{y^2 + 6^2}} = \frac{10^3}{4 \times 8.854} \ln(y + \sqrt{y^2 + 6^2})\Big|_3^5 = \underline{7.83\,\text{V}}$$

Line charge in an arc a radius $\rho = 3$:

$$V_{P2} = \int_{10^\circ}^{70^\circ} \frac{(1.5 \times 10^{-9})\,3\,d\phi}{4\pi\epsilon_0\sqrt{3^2 + 6^2}} = \frac{Q_2}{4\pi\epsilon_0\sqrt{45}} = \underline{6.31\,\text{V}}$$

Sheet charge:

$$V_{P3} = \int_{10^\circ}^{70^\circ} \int_{1.6}^{3.5} \frac{(10^{-9})\,\rho\,d\rho\,d\phi}{4\pi\epsilon_0\sqrt{\rho^2 + 6^2}} = \frac{60 \times 10^{-9}}{4\pi(8.854 \times 10^{-12}} \left(\frac{2\pi}{360}\right) \int_{1.6}^{3.5} \frac{\rho\,d\rho}{\sqrt{\rho^2 + 36}}$$
$$= 9.42\sqrt{\rho^2 + 36}\Big|_{1.6}^{3.5} = \underline{6.93\,\text{V}}$$

c) Find V_P: This will be the sum of the three results of part b, or

$$V_P = V_{P1} + V_{P2} + V_{P3} = 7.83 + 6.31 + 6.93 = \underline{21.1\,\text{V}}$$

4.21. Let $V = 2xy^2z^3 + 3\ln(x^2 + 2y^2 + 3z^2)$ V in free space. Evaluate each of the following quantities at $P(3, 2, -1)$:

a) V: Substitute P directly to obtain: $V = \underline{-15.0\,\text{V}}$

b) $|V|$. This will be just $\underline{15.0\,\text{V}}$.

c) $\mathbf{E}$: We have

$$\mathbf{E}\Big|_P = -\nabla V\Big|_P = -\left[\left(2y^2z^3 + \frac{6x}{x^2 + 2y^2 + 3z^2}\right)\mathbf{a}_x + \left(4xyz^3 + \frac{12y}{x^2 + 2y^2 + 3z^2}\right)\mathbf{a}_y\right.$$
$$\left. + \left(6xy^2z^2 + \frac{18z}{x^2 + 2y^2 + 3z^2}\right)\mathbf{a}_z\right]_P = \underline{7.1\mathbf{a}_x + 22.8\mathbf{a}_y - 71.1\mathbf{a}_z\ \text{V/m}}$$

4.21d. $|\mathbf{E}|_P$: taking the magnitude of the part c result, we find $|\mathbf{E}|_P = \underline{75.0\,\text{V/m}}$.

e) $\mathbf{a}_N$: By definition, this will be

$$\mathbf{a}_N\Big|_P = -\frac{\mathbf{E}}{|\mathbf{E}|} = \underline{-0.095\,\mathbf{a}_x - 0.304\,\mathbf{a}_y + 0.948\,\mathbf{a}_z}$$

f) $\mathbf{D}$: This is $\mathbf{D}\Big|_P = \epsilon_0 \mathbf{E}\Big|_P = \underline{62.8\,\mathbf{a}_x + 202\,\mathbf{a}_y - 629\,\mathbf{a}_z\ \text{pC/m}^2}$.

4.22. It is known that the potential is given as $V = 80r^{0.6}$ V. Assuming free space conditions, find:

a) $\mathbf{E}$: We use

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = -(0.6)80r^{-0.4}\,\mathbf{a}_r = \underline{-48r^{-0.4}\,\mathbf{a}_r\ \text{V/m}}$$

b) the volume charge density at $r = 0.5$ m: Begin by finding

$$\mathbf{D} = \epsilon_0 \mathbf{E} = -48r^{-0.4}\epsilon_0\,\mathbf{a}_r\ \text{C/m}^2$$

We next find

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 D_r\right) = \frac{1}{r^2}\frac{d}{dr}\left(-48\epsilon_0 r^{1.6}\right) = -\frac{76.8\epsilon_0}{r^{1.4}}\ \text{C/m}^3$$

Then at $r = 0.5$ m,

$$\rho_v(0.5) = \frac{-76.8(8.854 \times 10^{-12})}{(0.5)^{1.4}} = -1.79 \times 10^{-9}\ \text{C/m}^3 = \underline{-1.79\ \text{nC/m}^3}$$

c) the total charge lying within the surface $r = 0.6$: The easiest way is to use Gauss' law, and integrate the flux density over the spherical surface $r = 0.6$. Since the field is constant at constant radius, we obtain the product:

$$Q = 4\pi(0.6)^2(-48\epsilon_0(0.6)^{-0.4}) = -2.36 \times 10^{-9}\ \text{C} = \underline{-2.36\ \text{nC}}$$

4.23. It is known that the potential is given as $V = 80\rho^{.6}$ V. Assuming free space conditions, find:

a) $\mathbf{E}$: We find this through

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\mathbf{a}_\rho = \underline{-48\rho^{-.4}\,\text{V/m}}$$

b) the volume charge density at $\rho = .5$ m: Using $\mathbf{D} = \epsilon_0\mathbf{E}$, we find the charge density through

$$\rho_v\Big|_{.5} = [\nabla \cdot \mathbf{D}]_{.5} = \left(\frac{1}{\rho}\right)\frac{d}{d\rho}\left(\rho D_\rho\right)\Big|_{.5} = -28.8\epsilon_0\rho^{-1.4}\Big|_{.5} = \underline{-673\,\text{pC/m}^3}$$

4.23c. the total charge lying within the closed surface $\rho = .6$, $0 < z < 1$: The easiest way to do this calculation is to evaluate D_ρ at $\rho = .6$ (noting that it is constant), and then multiply by the cylinder area: Using part a, we have $D_\rho\Big|_{.6} = -48\epsilon_0(.6)^{-.4} = -521\,\text{pC/m}^2$. Thus $Q = -2\pi(.6)(1)521 \times 10^{-12}\,\text{C} = \underline{-1.96\,\text{nC}}$.

4.24. Given the potential field $V = 80r^2\cos\theta$ and a point $P(2.5, \theta = 30^\circ, \phi = 60^\circ)$ in free space, find at P:

a) V: Substitute the coordinates into the function and find $V_P = 80(2.5)^2\cos(30) = \underline{433\text{ V}}$.

b) $\mathbf{E}$:

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\mathbf{a}_r - \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta = -160r\cos\theta\,\mathbf{a}_r + 80r\sin\theta\,\mathbf{a}_\theta\ \text{V/m}$$

Evaluating this at P yields $\mathbf{E}_p = \underline{-346\mathbf{a}_r + 100\mathbf{a}_\theta\ \text{V/m}}$.

c) $\mathbf{D}$: In free space, $\mathbf{D}_P = \epsilon_0\mathbf{E}_P = (-346\mathbf{a}_r + 100\mathbf{a}_\theta)\epsilon_0 = \underline{-3.07\,\mathbf{a}_r + 0.885\,\mathbf{a}_\theta\ \text{nC/m}^2}$.

d) ρ_v:

$$\rho_v = \nabla\cdot\mathbf{D} = \epsilon_0\nabla\cdot\mathbf{E} = \epsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E_r\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(E_\theta\sin\theta\right)\right]$$

Substituting the components of $\mathbf{E}$, we find

$$\rho_v = \left[-\frac{160\cos\theta}{r^2}3r^2 + \frac{1}{r\sin\theta}80r(2\sin\theta\cos\theta)\right]\epsilon_0 = -320\epsilon_0\cos\theta = \underline{-2.45\text{ nC/m}^3}$$

with $\theta = 30^\circ$.

e) dV/dN: This will be just $|\mathbf{E}|$ evaluated at P, which is

$$\frac{dV}{dN}\Big|_P = |-346\mathbf{a}_r + 100\mathbf{a}_\theta| = \sqrt{(346)^2 + (100)^2} = \underline{360\text{ V/m}}$$

f) $\mathbf{a}_N$: This will be

$$\mathbf{a}_N = -\frac{\mathbf{E}_P}{|\mathbf{E}_P|} = -\left[\frac{-346\mathbf{a}_r + 100\mathbf{a}_\theta}{\sqrt{(346)^2 + (100)^2}}\right] = \underline{0.961\,\mathbf{a}_r - 0.278\,\mathbf{a}_\theta}$$

4.25. Within the cylinder $\rho = 2$, $0 < z < 1$, the potential is given by $V = 100 + 50\rho + 150\rho\sin\phi$ V.

a) Find V, $\mathbf{E}$, $\mathbf{D}$, and ρ_v at $P(1, 60^\circ, 0.5)$ in free space: First, substituting the given point, we find $V_P = \underline{279.9\text{ V}}$. Then,

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial\rho}\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial\phi}\mathbf{a}_\phi = -\left[50 + 150\sin\phi\right]\mathbf{a}_\rho - \left[150\cos\phi\right]\mathbf{a}_\phi$$

Evaluate the above at P to find $\mathbf{E}_P = \underline{-179.9\mathbf{a}_\rho - 75.0\mathbf{a}_\phi\ \text{V/m}}$

Now $\mathbf{D} = \epsilon_0\mathbf{E}$, so $\mathbf{D}_P = \underline{-1.59\mathbf{a}_\rho - .664\mathbf{a}_\phi\ \text{nC/m}^2}$. Then

$$\rho_v = \nabla\cdot\mathbf{D} = \left(\frac{1}{\rho}\right)\frac{d}{d\rho}\left(\rho D_\rho\right) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} = \left[-\frac{1}{\rho}(50 + 150\sin\phi) + \frac{1}{\rho}150\sin\phi\right]\epsilon_0 = -\frac{50}{\rho}\epsilon_0\ \text{C}$$

At P, this is $\rho_{vP} = \underline{-443\text{ pC/m}^3}$.

4.25b. How much charge lies within the cylinder? We will integrate ρ_v over the volume to obtain:

$$Q = \int_0^1 \int_0^{2\pi} \int_0^2 -\frac{50\epsilon_0}{\rho} \rho \, d\rho \, d\phi \, dz = -2\pi(50)\epsilon_0(2) = \underline{-5.56 \, \text{nC}}$$

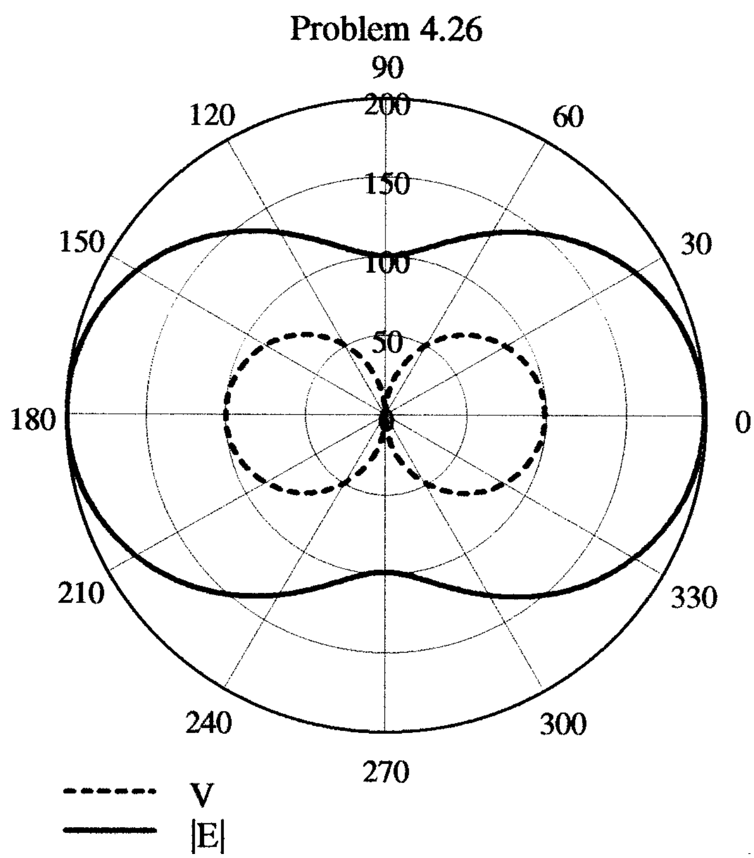
4.26. A dipole having $Qd/(4\pi\epsilon_0) = 100 \, \text{V} \cdot \text{m}^2$ is located at the origin in free space and aligned so that its moment is in the $\mathbf{a}_z$ direction. a) Sketch $|V(r = 1, \theta, \phi = 0)|$ versus θ on polar graph paper (homemade if you wish). b) Sketch $|\mathbf{E}(r = 1, \theta, \phi = 0)|$ versus θ on polar graph paper:

$$V = \frac{Qd \cos\theta}{4\pi\epsilon_0 r^2} = \frac{100 \cos\theta}{r^2} \quad \Rightarrow \quad |V(r = 1, \theta, \phi = 0)| = \underline{|100 \cos\theta|}$$

$$\mathbf{E} = \frac{Qd}{4\pi\epsilon_0 r^3} (2\cos\theta \, \mathbf{a}_r + \sin\theta \, \mathbf{a}_\theta) = \frac{100}{r^3} (2\cos\theta \, \mathbf{a}_r + \sin\theta \, \mathbf{a}_\theta)$$

$$|\mathbf{E}(r = 1, \theta, \phi = 0)| = 100 \left(4\cos^2\theta + \sin^2\theta\right)^{1/2} = \underline{100 \left(1 + 3\cos^2\theta\right)^{1/2}}$$

These results are plotted below:



4.27. Two point charges, 1 nC at (0, 0, 0.1) and −1 nC at (0, 0, −0.1), are in free space.

a) Calculate V at $P(0.3, 0, 0.4)$: Use

$$V_P = \frac{q}{4\pi\epsilon_0|\mathbf{R}^+|} - \frac{q}{4\pi\epsilon_0|\mathbf{R}^-|}$$

where $\mathbf{R}^+ = (.3, 0, .3)$ and $\mathbf{R}^- = (.3, 0, .5)$, so that $|\mathbf{R}^+| = 0.424$ and $|\mathbf{R}^-| = 0.583$. Thus

$$V_P = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{1}{.424} - \frac{1}{.583}\right] = \underline{5.78\,\text{V}}$$

b) Calculate $|\mathbf{E}|$ at P: Use

$$\mathbf{E}_P = \frac{q(.3\mathbf{a}_x + .3\mathbf{a}_z)}{4\pi\epsilon_0(.424)^3} - \frac{q(.3\mathbf{a}_x + .5\mathbf{a}_z)}{4\pi\epsilon_0(.583)^3} = \frac{10^{-9}}{4\pi\epsilon_0}\left[2.42\mathbf{a}_x + 1.41\mathbf{a}_z\right]\ \text{V/m}$$

Taking the magnitude of the above, we find $|\mathbf{E}_P| = \underline{25.2\,\text{V/m}}$.

c) Now treat the two charges as a dipole at the origin and find V at P: In spherical coordinates, P is located at $r = \sqrt{.3^2 + .4^2} = .5$ and $\theta = \sin^{-1}(.3/.5) = 36.9^\circ$. Assuming a dipole in far-field, we have

$$V_P = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{10^{-9}(.2)\cos(36.9^\circ)}{4\pi\epsilon_0(.5)^2} = \underline{5.76\,\text{V}}$$

4.28. A dipole located at the origin in free space has a moment $\mathbf{p}2 \times 10^{-9}\,\mathbf{a}_z$ C · m. At what points on the line $y = z,\ x = 0$ is:

a) $|E_\theta| = 1$ mV/m? We note that the line $y = z$ lies at $\theta = 45^\circ$. Begin with

$$\mathbf{E} = \frac{2\times 10^{-9}}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta) = \frac{10^{-9}}{2\sqrt{2}\pi\epsilon_0 r^3}(2\mathbf{a}_r + \mathbf{a}_\theta)\ \text{at}\ \theta = 45^\circ$$

from which

$$E_\theta = \frac{10^{-9}}{2\pi\epsilon_0 r^3} = 10^{-3}\ \text{V/m (required)} \quad\Rightarrow\quad r^3 = 1.27\times 10^{-4}\ \text{or}\ r = 23.3\,\text{m}$$

The y and z values are thus $y = z = \pm 23.3/\sqrt{2} = \underline{\pm 16.5\,\text{m}}$

b) $|E_r| = 1$ mV/m? From the above field expression, the radial component magnitude is twice that of the theta component. Using the same development, we then find

$$E_r = 2\frac{10^{-9}}{2\pi\epsilon_0 r^3} = 10^{-3}\ \text{V/m (required)} \quad\Rightarrow\quad r^3 = 2(1.27\times 10^{-4})\ \text{or}\ r = 29.4\,\text{m}$$

The y and z values are thus $y = z = \pm 29.4/\sqrt{2} = \underline{\pm 20.8\,\text{m}}$

4.29. A dipole having a moment $\mathbf{p} = 3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z$ nC · m is located at $Q(1, 2, -4)$ in free space. Find V at $P(2, 3, 4)$: We use the general expression for the potential in the far field:

$$V = \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = P - Q = (1, 1, 8)$. So

$$V_P = \frac{(3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z) \cdot (\mathbf{a}_x + \mathbf{a}_y + 8\mathbf{a}_z) \times 10^{-9}}{4\pi\epsilon_0 [1^2 + 1^2 + 8^2]^{1.5}} = \underline{1.31 \text{ V}}$$

4.30. A dipole, having a moment $\mathbf{p} = 2\mathbf{a}_z$ nC · m is located at the origin in free space. Give the magnitude of $\mathbf{E}$ and its direction $\mathbf{a}_E$ in cartesian components at $r = 100$ m, $\phi = 90^\circ$, and $\theta =$: a) 0°; b) 30°; c) 90°. Begin with

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right]$$

from which

$$|\mathbf{E}| = \frac{p}{4\pi\epsilon_0 r^3}\left[4\cos^2\theta + \sin^2\theta\right]^{1/2} = \frac{p}{4\pi\epsilon_0 r^3}\left[1 + 3\cos^2\theta\right]^{1/2}$$

Now

$$E_x = \mathbf{E} \cdot \mathbf{a}_x = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\mathbf{a}_r \cdot \mathbf{a}_x + \sin\theta\,\mathbf{a}_\theta \cdot \mathbf{a}_x\right] = \frac{p}{4\pi\epsilon_0 r^3}\left[3\cos\theta\sin\theta\cos\phi\right]$$

then

$$E_y = \mathbf{E} \cdot \mathbf{a}_y = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\mathbf{a}_r \cdot \mathbf{a}_y + \sin\theta\,\mathbf{a}_\theta \cdot \mathbf{a}_y\right] = \frac{p}{4\pi\epsilon_0 r^3}\left[3\cos\theta\sin\theta\sin\phi\right]$$

and

$$E_z = \mathbf{E} \cdot \mathbf{a}_z = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\mathbf{a}_r \cdot \mathbf{a}_z + \sin\theta\,\mathbf{a}_\theta \cdot \mathbf{a}_z\right] = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos^2\theta - \sin^2\theta\right]$$

Since ϕ is given as 90°, $E_x = 0$, and the field magnitude becomes

$$|\mathbf{E}(\phi = 90^\circ)| = \sqrt{E_y^2 + E_z^2} = \frac{p}{4\pi\epsilon_0 r^3}\left[9\cos^2\theta\sin^2\theta + (2\cos^2\theta - \sin^2\theta)^2\right]^{1/2}$$

Then the unit vector becomes (again at $\phi = 90^\circ$):

$$\mathbf{a}_E = \frac{3\cos\theta\sin\theta\,\mathbf{a}_y + (2\cos^2\theta - \sin^2\theta)\,\mathbf{a}_z}{\left[9\cos^2\theta\sin^2\theta + (2\cos^2\theta - \sin^2\theta)^2\right]^{1/2}}$$

Now with $r = 100$ m and $p = 2 \times 10^{-9}$,

$$\frac{p}{4\pi\epsilon_0 r^3} = \frac{2 \times 10^{-9}}{4\pi(8.854 \times 10^{-12})10^6} = 1.80 \times 10^{-5}$$

Using the above formulas, we find at $\theta = 0^\circ$, $|\mathbf{E}| = (1.80 \times 10^{-5})(2) = \underline{36.0\,\mu\text{V/m}}$ and $\mathbf{a}_E = \underline{\mathbf{a}_z}$. At $\theta = 30^\circ$, we find $|\mathbf{E}| = (1.80 \times 10^{-5})[1.69 + 1.56]^{1/2} = \underline{32.5\,\mu\text{V/m}}$ and $\mathbf{a}_E = (1.30\mathbf{a}_y + 1.25\mathbf{a}_z)/1.80 = \underline{0.72\,\mathbf{a}_x + 0.69\,\mathbf{a}_z}$. At $\theta = 90^\circ$, $|\mathbf{E}| = (1.80 \times 10^{-5})(1) = \underline{18.0\,\mu\text{V/m}}$ and $\mathbf{a}_E = \underline{-\mathbf{a}_z}$.

4.31. A potential field in free space is expressed as $V = 20/(xyz)$ V.

a) Find the total energy stored within the cube $1 < x, y, z < 2$. We integrate the energy density over the cube volume, where $w_E = (1/2)\epsilon_0 \mathbf{E} \cdot \mathbf{E}$, and where

$$\mathbf{E} = -\nabla V = 20 \left[\frac{1}{x^2yz} \mathbf{a}_x + \frac{1}{xy^2z} \mathbf{a}_y + \frac{1}{xyz^2} \mathbf{a}_z \right] \text{V/m}$$

The energy is now

$$W_E = 200\epsilon_0 \int_1^2 \int_1^2 \int_1^2 \left[\frac{1}{x^4y^2z^2} + \frac{1}{x^2y^4z^2} + \frac{1}{x^2y^2z^4} \right] dx\, dy\, dz$$

The integral evaluates as follows:

$$\begin{aligned}
W_E &= 200\epsilon_0 \int_1^2 \int_1^2 \left[-\left(\frac{1}{3}\right) \frac{1}{x^3y^2z^2} - \frac{1}{xy^4z^2} - \frac{1}{xy^2z^4} \right]_1^2 dy\, dz \\
&= 200\epsilon_0 \int_1^2 \int_1^2 \left[\left(\frac{7}{24}\right) \frac{1}{y^2z^2} + \left(\frac{1}{2}\right) \frac{1}{y^4z^2} + \left(\frac{1}{2}\right) \frac{1}{y^2z^4} \right] dy\, dz \\
&= 200\epsilon_0 \int_1^2 \left[-\left(\frac{7}{24}\right) \frac{1}{yz^2} - \left(\frac{1}{6}\right) \frac{1}{y^3z^2} - \left(\frac{1}{2}\right) \frac{1}{yz^4} \right]_1^2 dz \\
&= 200\epsilon_0 \int_1^2 \left[\left(\frac{7}{48}\right) \frac{1}{z^2} + \left(\frac{7}{48}\right) \frac{1}{z^2} + \left(\frac{1}{4}\right) \frac{1}{z^4} \right] dz \\
&= 200\epsilon_0 (3) \left[\frac{7}{96} \right] = \underline{387\,\text{pJ}}
\end{aligned}$$

b) What value would be obtained by assuming a uniform energy density equal to the value at the center of the cube? At $C(1.5, 1.5, 1.5)$ the energy density is

$$w_E = 200\epsilon_0 (3) \left[\frac{1}{(1.5)^4 (1.5)^2 (1.5)^2} \right] = 2.07 \times 10^{-10}\ \text{J/m}^3$$

This, multiplied by a cube volume of 1, produces an energy value of <u>207 pJ</u>.

4.32. In the region of free space where $2 < r < 3$, $0.4\pi < \theta < 0.6\pi$, $0 < \phi < \pi/2$, let $\mathbf{E} = k/r^2\, \mathbf{a}_r$.

a) Find a positive value for k so that the total energy stored is exactly 1 J: The energy is found through

$$\begin{aligned}
W_E &= \int_v \frac{1}{2}\epsilon_0 E^2\, dv = \int_0^{\pi/2} \int_{0.4\pi}^{0.6\pi} \int_2^3 \frac{1}{2}\epsilon_0 \frac{k^2}{r^2} r^2 \sin\theta\, dr\, d\theta\, d\phi \\
&= \frac{\pi}{2} (-\cos\theta) \Big|_{.4\pi}^{.6\pi} \left(\frac{1}{2}\right) \epsilon_0 k^2 \left(-\frac{1}{r}\right) \Big|_2^3 = \frac{0.616\pi}{24} \epsilon_0 k^2 = 1\,\text{J}
\end{aligned}$$

Solve for k to find $k = \underline{1.18 \times 10^6\ \text{V} \cdot \text{m}}$.

4.32b. Show that the surface $\theta = 0.6\pi$ is an equipotential surface: This will be the surface of a cone, centered at the origin, along which **E**, in the $\mathbf{a}_r$ direction, will exist. Therefore, the given surface *cannot* be an equipotential (the problem was ill-conceived). Only a surface of constant r could be an equipotential in this field.

c) Find V_{AB}, given points $A(2, \theta = \pi/2, \phi = \pi/3)$ and $B(3, \pi/2, \pi/4)$: Use

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_2^3 \frac{k}{r^2}\mathbf{a}_r \cdot \mathbf{a}_r \, dr = k\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{k}{6}$$

Using the result of part a, we find $V_{AB} = (1.18 \times 10^6)/6 = \underline{197 \text{ kV}}$.

4.33. A copper sphere of radius 4 cm carries a uniformly-distributed total charge of 5 μC in free space.

a) Use Gauss' law to find **D** external to the sphere: with a spherical Gaussian surface at radius r, D will be the total charge divided by the area of this sphere, and will be $\mathbf{a}_r$-directed. Thus

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r = \frac{5 \times 10^{-6}}{4\pi r^2}\mathbf{a}_r \text{ C/m}^2$$

b) Calculate the total energy stored in the electrostatic field: Use

$$W_E = \int_{vol} \frac{1}{2}\mathbf{D} \cdot \mathbf{E} \, dv = \int_0^{2\pi} \int_0^{\pi} \int_{.04}^{\infty} \frac{1}{2} \frac{(5 \times 10^{-6})^2}{16\pi^2 \epsilon_0 r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi$$
$$= (4\pi)\left(\frac{1}{2}\right) \frac{(5 \times 10^{-6})^2}{16\pi^2 \epsilon_0} \int_{.04}^{\infty} \frac{dr}{r^2} = \frac{25 \times 10^{-12}}{8\pi\epsilon_0} \frac{1}{.04} = \underline{2.81 \text{ J}}$$

c) Use $W_E = Q^2/(2C)$ to calculate the capacitance of the isolated sphere: We have

$$C = \frac{Q^2}{2W_E} = \frac{(5 \times 10^{-6})^2}{2(2.81)} = 4.45 \times 10^{-12} \text{ F} = \underline{4.45 \text{ pF}}$$

4.34. Given the potential field in free space, $V = 80\phi$ V (note that $\mathbf{a}_phi$ should not be present), find:

a) the energy stored in the region $2 < \rho < 4$ cm, $0 < \phi < 0.2\pi$, $0 < z < 1$ m: First we find

$$\mathbf{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\mathbf{a}_\phi = -\frac{80}{\rho}\mathbf{a}_\phi \text{ V/m}$$

Then

$$W_E = \int_v w_E dv = \int_0^1 \int_0^{0.2\pi} \int_{.02}^{.04} \frac{1}{2}\epsilon_0 \frac{(80)^2}{\rho^2} \rho \, d\rho \, d\phi \, dz = 640\pi\epsilon_0 \ln\left(\frac{.04}{.02}\right) = \underline{12.3 \text{ nJ}}$$

b) the potential difference, V_{AB}, for $A(3 \text{ cm}, \phi = 0, z = 0)$ and $B(3\text{cm}, 0.2\pi, 1\text{m})$: Use

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_{.2\pi}^0 -\frac{80}{\rho} \mathbf{a}_\phi \cdot \mathbf{a}_\phi \, \rho \, d\phi = -80(0.2\pi) = \underline{-16\pi \text{ V}}$$

4.34c. the maximum value of the energy density in the specified region: The energy density is

$$w_E = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\epsilon_0 \frac{6400}{\rho^2}$$

This will maximize at the lowest value of ρ in the specified range, which is $\rho = 2$ cm. So

$$w_{E,max} = \frac{1}{2}\epsilon_0 \frac{6400}{.02^2} = 7.1 \times 10^{-5} \text{ J/m}^3 = \underline{71\ \mu\text{J/m}^3}$$

4.35. Four 0.8 nC point charges are located in free space at the corners of a square 4 cm on a side.

a) Find the total potential energy stored: This will be given by

$$W_E = \frac{1}{2}\sum_{n=1}^{4} q_n V_n$$

where V_n in this case is the potential at the location of any one of the point charges that arises from the other three. This will be (for charge 1)

$$V_1 = V_{21} + V_{31} + V_{41} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{.04} + \frac{1}{.04} + \frac{1}{.04\sqrt{2}}\right]$$

Taking the summation produces a factor of 4, since the situation is the same at all four points. Consequently,

$$W_E = \frac{1}{2}(4)q_1 V_1 = \frac{(.8 \times 10^{-9})^2}{2\pi\epsilon_0(.04)}\left[2 + \frac{1}{\sqrt{2}}\right] = 7.79 \times 10^{-7} \text{ J} = \underline{0.779\,\mu\text{J}}$$

b) A fifth 0.8 μC charge is installed at the center of the square. Again find the total stored energy: This will be the energy found in part *a* plus the amount of work done in moving the fifth charge into position from infinity. The latter is just the potential at the square center arising from the original four charges, times the new charge value, or

$$\Delta W_E = \frac{4(.8 \times 10^{-9})^2}{4\pi\epsilon_0(.04\sqrt{2}/2)} = .813\,\mu\text{J}$$

The total energy is now

$$W_{E\,net} = W_E(\text{part a}) + \Delta W_E = .779 + .813 = \underline{1.59\,\mu\text{J}}$$