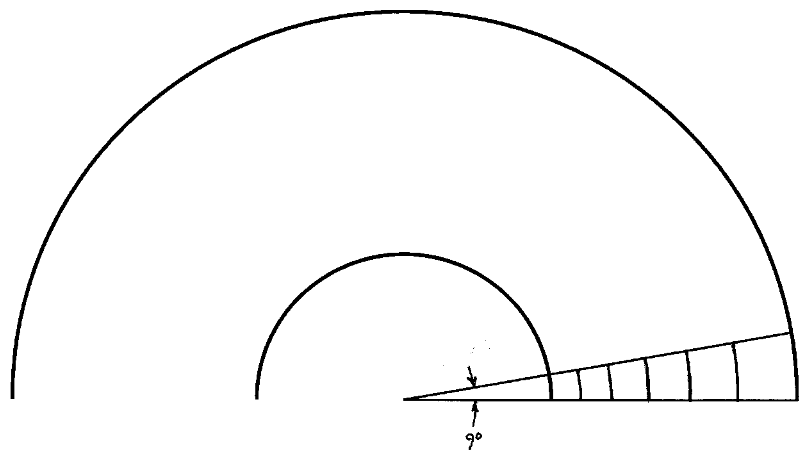
CHAPTER 6.

6.1 Construct a curvilinear square map for a coaxial capacitor of 3-cm inner radius and 8-cm outer radius. These dimensions are suitable for the drawing.

a) Use your sketch to calculate the capacitance per meter length, assuming $\epsilon_R = 1$: The sketch is shown below. Note that only a 9° sector was drawn, since this would then be duplicated 40 times around the circumference to complete the drawing. The capacitance is thus

$$C \doteq \epsilon_0 \frac{N_Q}{N_V} = \epsilon_0 \frac{40}{6} = \underline{59 \text{ pF/m}}$$



b) Calculate an exact value for the capacitance per unit length: This will be

$$C = \frac{2\pi\epsilon_0}{\ln(8/3)} = \underline{57 \text{ pF/m}}$$

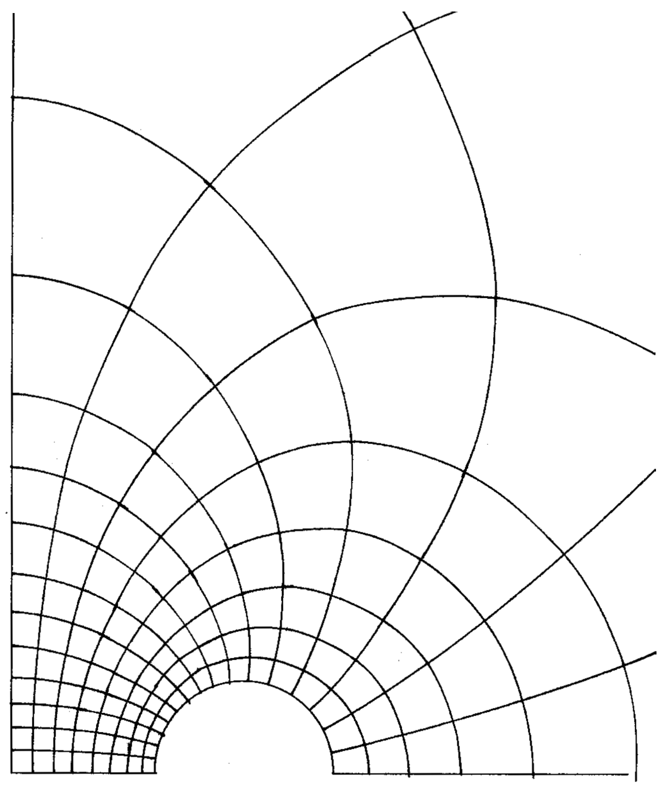
6.2 Construct a curvilinear-square map of the potential field about two parallel circular cylinders, each of 2.5 cm radius, separated by a center-to-center distance of 13cm. These dimensions are suitable for the actual sketch if symmetry is considered. As a check, compute the capacitance per meter both from your sketch and from the exact formula. Assume $\epsilon_R = 1$.

Symmetry allows us to plot the field lines and equipotentials over just the first quadrant, as is done in the sketch below (shown to one-half scale). The capacitance is found from the formula $C = (N_Q/N_V)\epsilon_0$, where N_Q is twice the number of squares around the perimeter of the half-circle and N_V is twice the number of squares between the half-circle and the left vertical plane. The result is

$$C = \frac{N_Q}{N_V}\epsilon_0 = \frac{32}{16}\epsilon_0 = 2\epsilon_0 = \underline{17.7 \text{ pF/m}}$$

We check this result with that using the exact formula:

$$C = \frac{\pi\epsilon_0}{\cosh^{-1}(d/2a)} = \frac{\pi\epsilon_0}{\cosh^{-1}(13/5)} = 1.95\epsilon_0 = \underline{17.3 \text{ pF/m}}$$



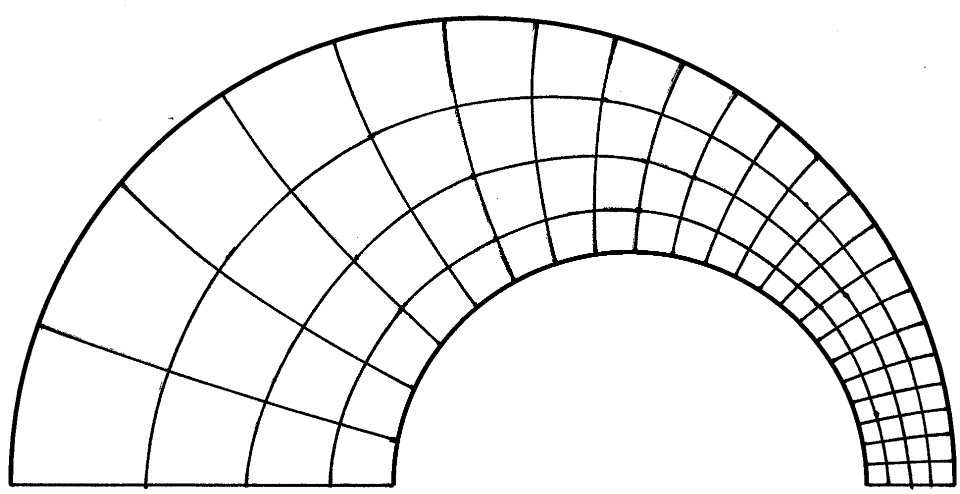
6.3. Construct a curvilinear square map of the potential field between two parallel circular cylinders, one of 4-cm radius inside one of 8-cm radius. The two axes are displaced by 2.5 cm. These dimensions are suitable for the drawing. As a check on the accuracy, compute the capacitance per meter from the sketch and from the exact expression:

$$C = \frac{2\pi\epsilon}{\cosh^{-1}\left[(a^2 + b^2 - D^2)/(2ab)\right]}$$

where a and b are the conductor radii and D is the axis separation.

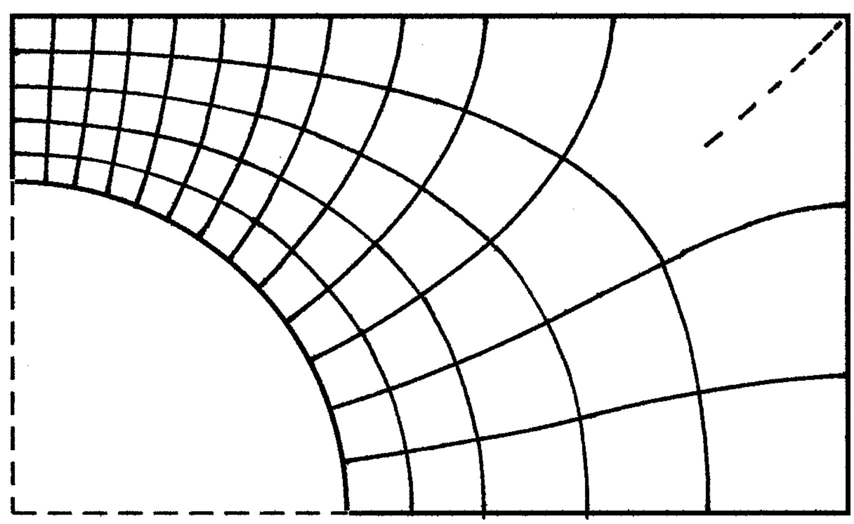
The drawing is shown below. Use of the exact expression above yields a capacitance value of $C = \underline{11.5\epsilon_0 \text{ F/m}}$. Use of the drawing produces:

$$C \doteq \frac{22 \times 2}{4}\epsilon_0 = \underline{11\epsilon_0 \text{ F/m}}$$



6.4. A solid conducting cylinder of 4-cm radius is centered within a rectangular conducting cylinder with a 12-cm by 20-cm cross-section.

a) Make a full-size sketch of one quadrant of this configuration and construct a curvilinear-square map for its interior: The result below could still be improved a little, but is nevertheless sufficient for a reasonable capacitance estimate. Note that the five-sided region in the upper right corner has been partially subdivided (dashed line) in anticipation of how it would look when the next-level subdivision is done (doubling the number of field lines and equipotentials).

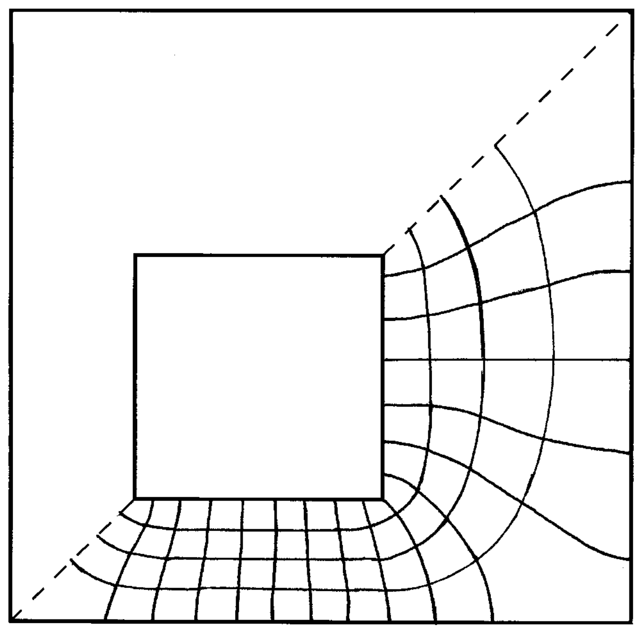


b) Assume $\epsilon = \epsilon_0$ and estimate C per meter length: In this case N_Q is the number of squares around the full perimeter of the circular conductor, or four times the number of squares shown in the drawing. N_V is the number of squares between the circle and the rectangle, or 5. The capacitance is estimated to be

$$C = \frac{N_Q}{N_V}\epsilon_0 = \frac{4 \times 13}{5}\epsilon_0 = 10.4\epsilon_0 \doteq \underline{90 \text{ pF/m}}$$

6.5. The inner conductor of the transmission line shown in Fig. 6.12 has a square cross-section $2a \times 2a$, while the outer square is $5a \times 5a$. The axes are displaced as shown. (a) Construct a good-sized drawing of the transmission line, say with $a = 2.5$ cm, and then prepare a curvilinear-square plot of the electrostatic field between the conductors. (b) Use the map to calculate the capacitance per meter length if $\epsilon = 1.6\epsilon_0$. (c) How would your result to part b change if $a = 0.6$ cm?

a) The plot is shown below. Some improvement is possible, depending on how much time one wishes to spend.



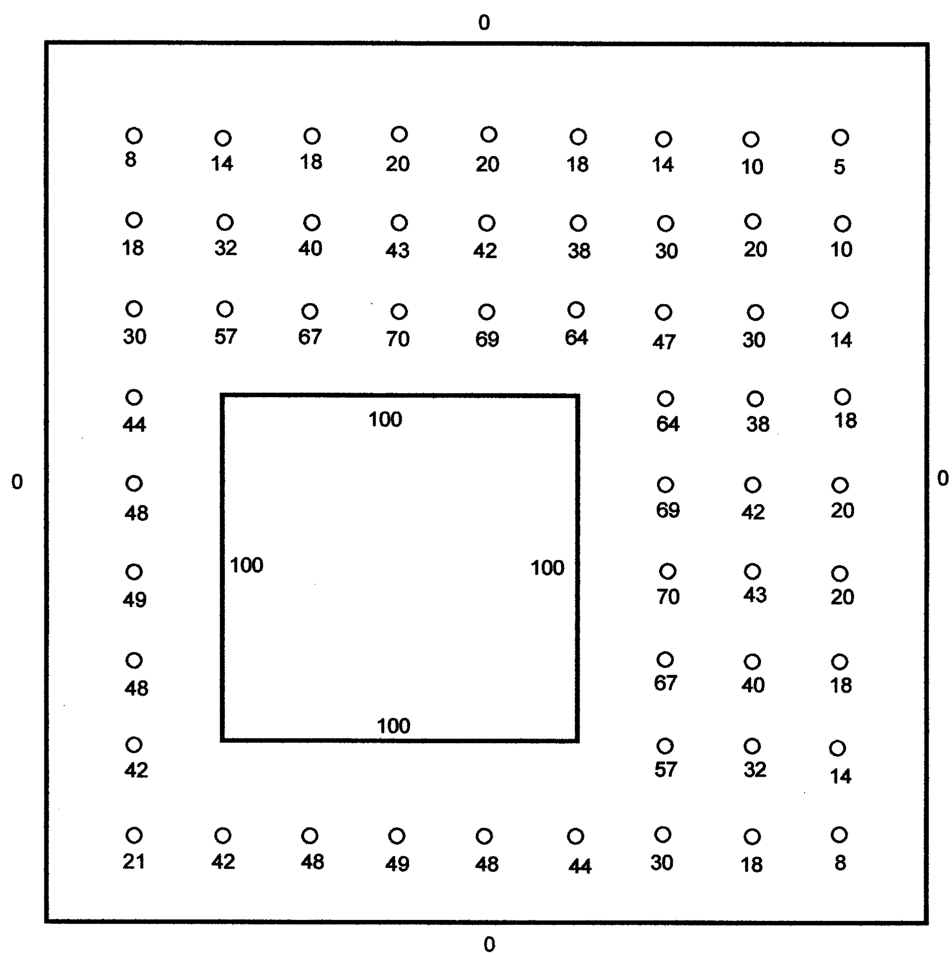
b) From the plot, the capacitance is found to be

$$C \doteq \frac{16 \times 2}{4}(1.6)\epsilon_0 = 12.8\epsilon_0 \doteq \underline{110 \text{ pF/m}}$$

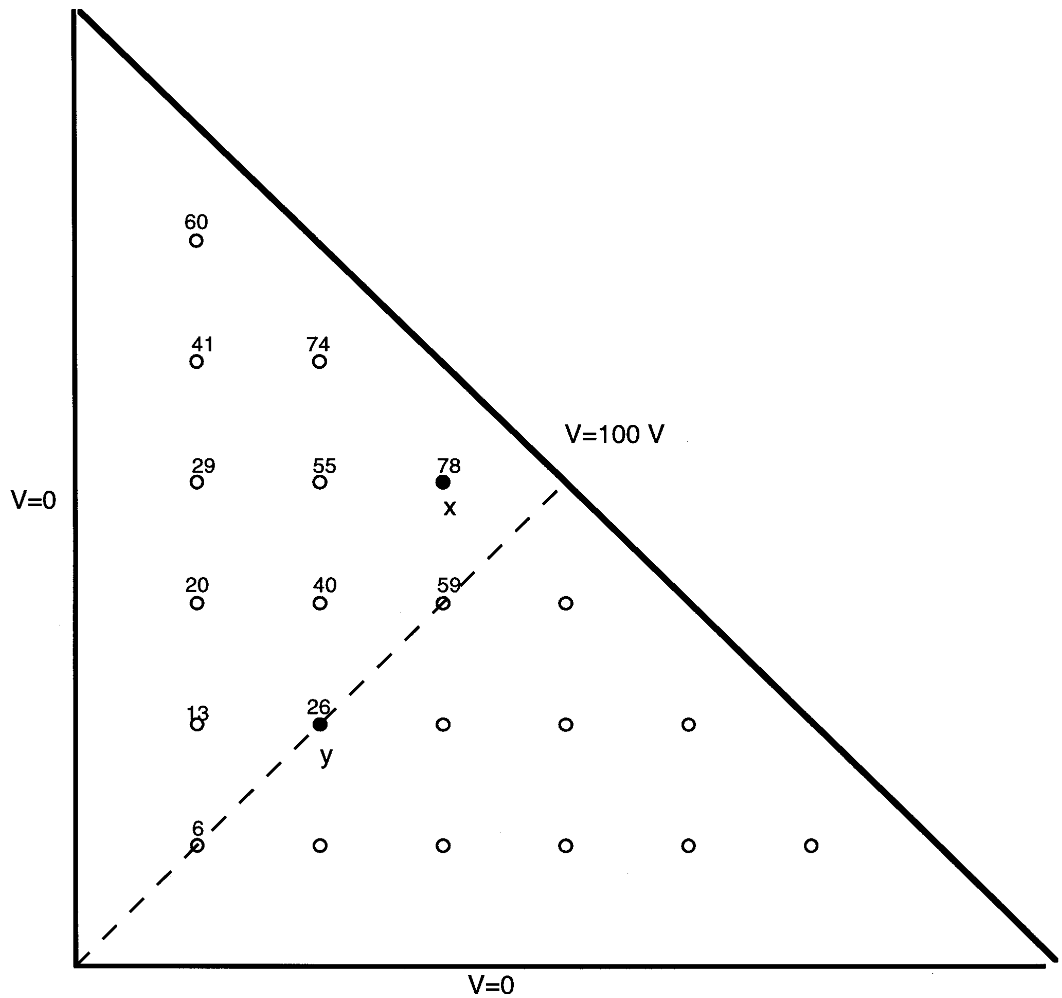
c) If a is changed, the result of part b would not change, since all dimensions retain the same relative scale.

6.6. Let the inner conductor of the transmission line shown in Fig. 6.12 be at a potential of 100V, while the outer is at zero potential. Construct a grid, $0.5a$ on a side, and use iteration to find V at a point that is a units above the upper right corner of the inner conductor. Work to the nearest volt:

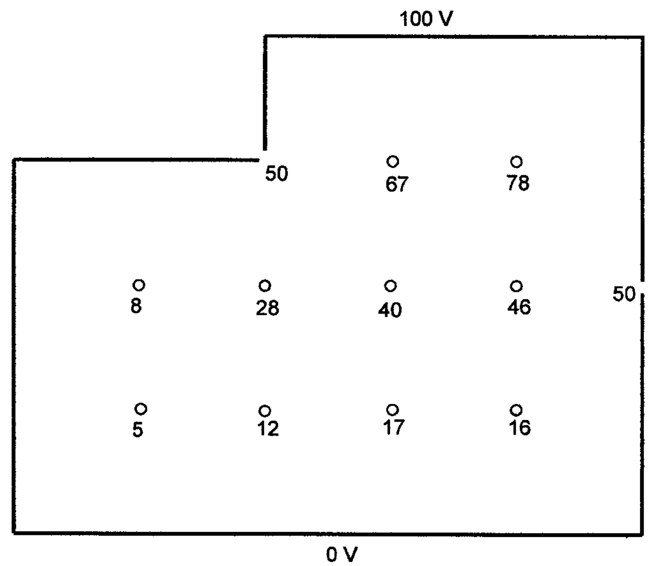
The drawing is shown below, and we identify the requested voltage as 38 V.



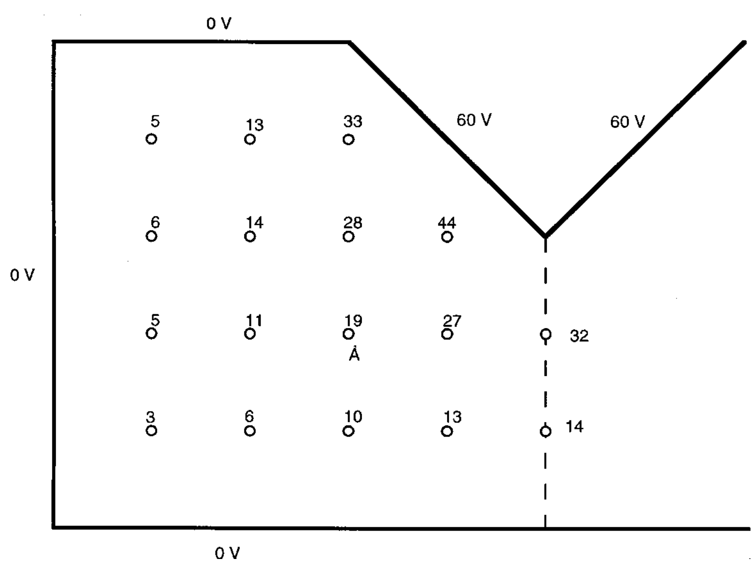
6.7. Use the iteration method to estimate the potentials at points x and y in the triangular trough of Fig. 6.13. Work only to the nearest volt: The result is shown below. The mirror image of the values shown occur at the points on the other side of the line of symmetry (dashed line). Note that $V_x = \underline{78\,\text{V}}$ and $V_y = \underline{26\,\text{V}}$.



6.8. Use iteration methods to estimate the potential at point x in the trough shown in Fig. 6.14. Working to the nearest volt is sufficient. The result is shown below, where we identify the voltage at x to be <u>40 V</u>. Note that the potentials in the gaps are 50 V.



6.9. Using the grid indicated in Fig. 6.15, work to the nearest volt to estimate the potential at point A: The voltages at the grid points are shown below, where V_A is found to be <u>19 V</u>. Half the figure is drawn since mirror images of all values occur across the line of symmetry (dashed line).



6.10. Conductors having boundaries that are curved or skewed usually do not permit every grid point to coincide with the actual boundary. Figure 6.16a illustrates the situation where the potential at V_0 is to be estimated in terms of V_1, V_2, V_3, and V_4, and the unequal distances h_1, h_2, h_3, and h_4.

a) Show that

$$V_0 \doteq \frac{V_1}{\left(1+\frac{h_1}{h_3}\right)\left(1+\frac{h_1 h_3}{h_4 h_2}\right)} + \frac{V_2}{\left(1+\frac{h_2}{h_4}\right)\left(1+\frac{h_2 h_4}{h_1 h_3}\right)} + \frac{V_3}{\left(1+\frac{h_3}{h_1}\right)\left(1+\frac{h_1 h_3}{h_4 h_2}\right)}$$
$$+ \frac{V_4}{\left(1+\frac{h_4}{h_2}\right)\left(1+\frac{h_4 h_2}{h_3 h_1}\right)} \quad \text{note error, corrected here, in the equation (second term)}$$

Referring to the figure, we write:

$$\left.\frac{\partial V}{\partial x}\right|_{M_1} \doteq \frac{V_1 - V_0}{h_1} \qquad\qquad \left.\frac{\partial V}{\partial x}\right|_{M_3} \doteq \frac{V_0 - V_3}{h_3}$$

Then

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{V_0} \doteq \frac{(V_1 - V_0)/h_1 - (V_0 - V_3)/h_3}{(h_1 + h_3)/2} = \frac{2V_1}{h_1(h_1 + h_3)} + \frac{2V_3}{h_3(h_1 + h_3)} - \frac{2V_0}{h_1 h_3}$$

We perform the same procedure along the y axis to obtain:

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_{V_0} \doteq \frac{(V_2 - V_0)/h_2 - (V_0 - V_4)/h_4}{(h_2 + h_4)/2} = \frac{2V_2}{h_2(h_2 + h_4)} + \frac{2V_4}{h_4(h_2 + h_4)} - \frac{2V_0}{h_2 h_4}$$

Then, knowing that

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{V_0} + \left.\frac{\partial^2 V}{\partial y^2}\right|_{V_0} = 0$$

the two equations for the second derivatives are added to give

$$\frac{2V_1}{h_1(h_1 + h_3)} + \frac{2V_2}{h_2(h_2 + h_4)} + \frac{2V_3}{h_3(h_1 + h_3)} + \frac{2V_4}{h_4(h_2 + h_4)} = V_0\left(\frac{h_1 h_3 + h_2 h_4}{h_1 h_2 h_3 h_4}\right)$$

Solve for V_0 to obtain the given equation.

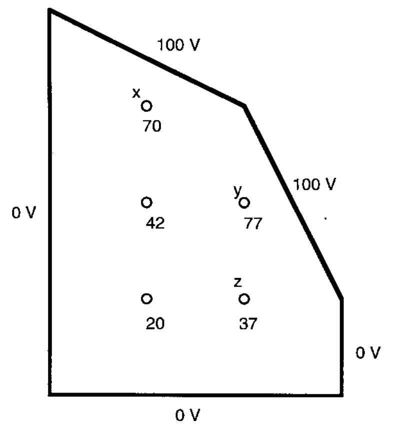
b) Determine V_0 in Fig. 6.16b: Referring to the figure, we note that $h_1 = h_2 = a$. The other two distances are found by writing equations for the circles:

$$(0.5a + h_3)^2 + a^2 = (1.5a)^2 \;\text{ and }\; (a + h_4)^2 + (0.5a)^2 = (1.5a)^2$$

These are solved to find $h_3 = 0.618a$ and $h_4 = 0.414a$. The four distances and potentials are now substituted into the given equation:

$$V_0 \doteq \frac{80}{\left(1+\frac{1}{.618}\right)\left(1+\frac{.618}{.414}\right)} + \frac{60}{\left(1+\frac{1}{.414}\right)\left(1+\frac{.414}{.618}\right)} + \frac{100}{(1+.618)\left(1+\frac{.618}{.414}\right)}$$
$$+ \frac{100}{(1+.414)\left(1+\frac{.414}{.618}\right)} = \underline{90\ \text{V}}$$

6.11. Consider the configuration of conductors and potentials shown in Fig. 6.17. Using the method described in Problem 10, write an expression for V_x (not V_0): The result is shown below, where $V_x = \underline{70\,\text{V}}$.



6.12a) After estimating potentials for the configuation of Fig. 6.18, use the iteration method with a square grid 1 cm on a side to find better estimates at the seven grid points. Work to the nearest volt:

25	50	75	50	25
0	48	100	48	0
0	42	100	42	0
0	19	34	19	0
0	0	0	0	0

b) Construct a 0.5 cm grid, establish new rough estimates, and then use the iteration method on the 0.5 cm grid. Again, work to the nearest volt: The result is shown below, with values for the original grid points underlined:

25	50	50	50	75	50	50	50	25
0	32	50	68	100	68	50	32	0
0	26	48	72	100	72	48	26	0
0	23	45	70	100	70	45	23	0
0	20	40	64	100	64	40	20	0
0	15	30	44	54	44	30	15	0
0	10	19	26	30	26	19	10	0
0	5	9	12	14	12	9	5	0
0	0	0	0	0	0	0	0	0

6.12c. Use the computer to obtain values for a 0.25 cm grid. Work to the nearest 0.1 V: Values for the left half of the configuration are shown in the table below. Values along the vertical line of symmetry are included, and the original grid values are underlined.

25	50	50	50	50	50	50	50	75
0	26.5	38.0	44.6	49.6	54.6	61.4	73.2	100
0	18.0	31.0	40.7	49.0	57.5	67.7	81.3	100
0	14.5	27.1	38.1	48.3	58.8	70.6	84.3	100
0	12.8	24.8	36.2	<u>47.3</u>	58.8	71.4	85.2	100
0	11.7	23.1	34.4	45.8	57.8	70.8	85.0	100
0	10.8	21.6	32.5	43.8	55.8	69.0	83.8	100
0	10.0	20.0	30.2	40.9	52.5	65.6	81.2	100
0	9.0	18.1	27.4	<u>37.1</u>	47.6	59.7	75.2	100
0	7.9	15.9	24.0	32.4	41.2	50.4	59.8	67.2
0	6.8	13.6	20.4	27.3	34.2	40.7	46.3	49.2
0	5.6	11.2	16.8	22.2	27.4	32.0	35.4	36.8
0	4.4	8.8	13.2	<u>17.4</u>	21.2	24.4	26.6	<u>27.4</u>
0	3.3	6.6	9.8	12.8	15.4	17.6	19.0	19.5
0	2.2	4.4	6.4	8.4	10.0	11.4	12.2	12.5
0	1.1	2.2	3.2	4.2	5.0	5.6	6.0	6.1
0	0	0	0	0	0	0	0	0

6.13. Perfectly-conducting concentric spheres have radii of 2 and 6 cm. The region $2 < r < 3$ cm is filled with a solid conducting material for which $\sigma = 100$ S/m, while the portion for which $3 < r < 6$ cm has $\sigma = 25$ S/m. The inner sphere is held at 1 V while the outer is at $V = 0$.

a. Find E and J everywhere: From symmetry, E and J will be radially-directed, and we note the fact that the current, I, must be constant at any cross-section; i.e., through any spherical surface at radius r between the spheres. Thus we require that in both regions,

$$\mathbf{J} = \frac{I}{4\pi r^2}\mathbf{a}_r$$

The fields will thus be

$$\mathbf{E}_1 = \frac{I}{4\pi\sigma_1 r^2}\mathbf{a}_r \ \ (2 < r < 3) \quad \text{and} \quad \mathbf{E}_2 = \frac{I}{4\pi\sigma_2 r^2}\mathbf{a}_r \ \ (3 < r < 6)$$

where $\sigma_1 = 100$ S/m and $\sigma_2 = 25$ S/m. Since we know the voltage between spheres (1V), we can find the value of I through:

$$1\,\text{V} = -\int_{.06}^{.03} \frac{I}{4\pi\sigma_2 r^2}\,dr - \int_{.03}^{.02} \frac{I}{4\pi\sigma_1 r^2}\,dr = \frac{I}{0.24\pi}\left[\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right]$$

and so

$$I = \frac{0.24\pi}{(1/\sigma_1 + 1/\sigma_2)} = 15.08\,\text{A}$$

Then finally, with $I = 15.08$ A substituted into the field expressions above, we find

$$\mathbf{E}_1 = \underline{\frac{.012}{r^2}\,\mathbf{a}_r \ \text{V/m}} \ \ (2 < r < 3)$$

and

$$\mathbf{E}_2 = \underline{\frac{.048}{r^2}\,\mathbf{a}_r \ \text{V/m}} \ \ (3 < r < 6)$$

The current density is now

$$\mathbf{J} = \sigma_1\mathbf{E}_1 = \sigma_2\mathbf{E}_2 = \underline{\frac{1.2}{r^2}\ \text{A/m}} \ \ (2 < r < 6)$$

b) What resistance would be measured between the two spheres? We use

$$R = \frac{V}{I} = \frac{1\,\text{V}}{15.08\,\text{A}} = \underline{6.63 \times 10^{-2}\ \Omega}$$

c) What is V at $r = 3$ cm? This we find through

$$V = -\int_{.06}^{.03} \frac{.048}{r^2}\,dr = .048\left(\frac{1}{.03} - \frac{1}{.06}\right) = \underline{0.8\ \text{V}}$$

6.14. The cross-section of the transmission line shown in Fig. 6.12 is drawn on a sheet of conducting paper with metallic paint. The sheet resistance is 2000 Ω/sq and the dimension a is 2 cm.

a) Assuming a result for Prob. 6b of 110 pF/m, what total resistance would be measured between the metallic conductors drawn on the conducting paper? We assume a paper thickness of t m, so that the capacitance is $C = 110t$ pF, and the surface resistance is $R_s = 1/(\sigma t) = 2000\ \Omega/\text{sq}$. We now use

$$RC = \frac{\epsilon}{\sigma} \Rightarrow R = \frac{\epsilon}{\sigma C} = \frac{\epsilon R_s t}{110 \times 10^{-12} t} = \frac{(1.6 \times 8.854 \times 10^{-12})(2000)}{110 \times 10^{-12}} = \underline{257.6\ \Omega}$$

b) What would the total resistance be if $a = 2$ cm? The result is independent of a, provided the proportions are maintained. So again, $R = \underline{257.6\ \Omega}$.

6.15. two concentric annular rings are painted on a sheet of conducting paper with a highly conducting metal paint. The four radii are 1, 1.2, 3.5, and 3.7 cm. Connections made to the two rings show a resistance of 215 ohms between them.

a) What is R_s for the conducting paper? Using the two radii (1.2 and 3.5 cm) at which the rings are at their closest separation, we first evaluate the capacitance:

$$C = \frac{2\pi\epsilon_0 t}{\ln(3.5/1.2)} = 5.19 \times 10^{-11} t \text{ F}$$

where t is the unknown paper coating thickness. Now use

$$RC = \frac{\epsilon_0}{\sigma} \Rightarrow R = \frac{8.85 \times 10^{-12}}{5.19 \times 10^{-11} \sigma t} = 215$$

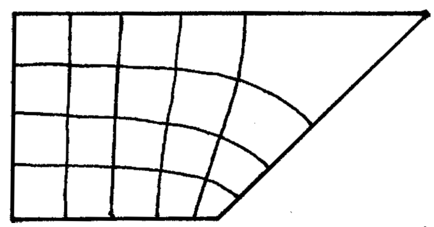
Thus

$$R_s = \frac{1}{\sigma t} = \frac{(51.9)(215)}{8.85} = \underline{1.26\,\text{k}\Omega/\text{sq}}$$

b) If the conductivity of the material used as the surface of the paper is 2 S/m, what is the thickness of the coating? We use

$$t = \frac{1}{\sigma R_s} = \frac{1}{2 \times 1.26 \times 10^3} = 3.97 \times 10^{-4} \text{ m} = \underline{0.397 \text{ mm}}$$

6.16. The square washer shown in Fig. 6.19 is 2.4 mm thick and has outer dimensions of 2.5 × 2.5 cm and inner dimensions of 1.25 × 1.25 cm. The inside and outside surfaces are perfectly-conducting. If the material has a conductivity of 6 S/m, estimate the resistance offered between the inner and outer surfaces (shown shaded in Fig. 6.19). A few curvilinear squares are suggested: First we find the surface resistance, $R_s = 1/(\sigma t) = 1/(6 \times 2.4 \times 10^{-3}) = 69.4\ \Omega/\text{sq}$. Having found this, we can construct the total resistance by using the fundamental square as a building block. Specifically, $R = R_s(N_l/N_w)$ where N_l is the number of squares between the inner and outer surfaces and N_w is the number of squares around the perimeter of the washer. These numbers are found from the curvilinear square plot shown below, which covers one-eighth the washer. The resistance is thus $R \doteq 69.4[4/(8 \times 5)] \doteq \underline{6.9\ \Omega}$.



6.17. A two-wire transmission line consists of two parallel perfectly-conducting cylinders, each having a radius of 0.2 mm, separated by center-to-center distance of 2 mm. The medium surrounding the wires has $\epsilon_R = 3$ and $\sigma = 1.5$ mS/m. A 100-V battery is connected between the wires. Calculate:

a) the magnitude of the charge per meter length on each wire: Use

$$C = \frac{\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{\pi \times 3 \times 8.85 \times 10^{-12}}{\cosh^{-1}(1/0.2)} = 3.64 \times 10^{-9}\ \text{C/m}$$

Then the charge per unit length will be

$$Q = CV_0 = (3.64 \times 10^{-11})(100) = 3.64 \times 10^{-9}\ \text{C/m} = \underline{3.64\,\text{nC/m}}$$

b) the battery current: Use

$$RC = \frac{\epsilon}{\sigma} \quad \Rightarrow \quad R = \frac{3 \times 8.85 \times 10^{-12}}{(1.5 \times 10^{-3})(3.64 \times 10^{-11})} = 486\,\Omega$$

Then

$$I = \frac{V_0}{R} = \frac{100}{486} = 0.206\,\text{A} = \underline{206\,\text{mA}}$$

6.18. A coaxial transmission line is modelled by the use of a rubber sheet having horizontal dimensions that are 100 times those of the actual line. Let the radial coordinate of the model be ρ_m. For the line itself, let the radial dimension be designated by ρ as usual; also, let $a = 0.6$ mm and $b = 4.8$ mm. The model is 8 cm in height at the inner conductor and zero at the outer. If the potential of the inner conductor is 100 V:

a) Find the expression for $V(\rho)$: Assuming charge density ρ_s on the inner conductor, we use Gauss' Law to find $2\pi\rho D = 2\pi a\rho_s$, from which $E = D/\epsilon = a\rho_s/(\epsilon\rho)$ in the radial direction. The potential difference between inner and outer conductors is

$$V_{ab} = V_0 = -\int_b^a \frac{a\rho_s}{\epsilon\rho}\,d\rho = \frac{a\rho_s}{\epsilon}\ln\left(\frac{b}{a}\right)$$

from which

$$\rho_s = \frac{\epsilon V_0}{a\ln(b/a)} \quad \Rightarrow \quad E = \frac{V_0}{\rho\ln(b/a)}$$

Now, as a function of radius, and assuming zero potential on the outer conductor, the potential function will be:

$$V(\rho) = -\int_b^\rho \frac{V_0}{\rho'\ln(b/a)}\,d\rho' = V_0\frac{\ln(b/\rho)}{\ln(b/a)} = 100\frac{\ln(.0048/\rho)}{\ln(.0048/.0006)} = \underline{48.1\ln\left(\frac{.0048}{\rho}\right)\ \text{V}}$$

b) Write the model height as a function of ρ_m (not ρ): We use the part a result, since the gravitational function must be the same as that for the electric potential. We replace V_0 by the maximum height, and multiply all dimensions by 100 to obtain:

$$h(\rho_m) = 0.08\frac{\ln(.48/\rho_m)}{\ln(.48/.06)} = \underline{0.038\ln\left(\frac{.48}{\rho_m}\right)\ \text{m}}$$