CHAPTER 9

9.1. A point charge, $Q = -0.3\,\mu$C and $m = 3 \times 10^{-16}$ kg, is moving through the field $\mathbf{E} = 30\,\mathbf{a}_z$ V/m. Use Eq. (1) and Newton's laws to develop the appropriate differential equations and solve them, subject to the initial conditions at $t = 0$: $\mathbf{v} = 3 \times 10^5\,\mathbf{a}_x$ m/s at the origin. At $t = 3\,\mu$s, find:

a) the position $P(x, y, z)$ of the charge: The force on the charge is given by $\mathbf{F} = q\mathbf{E}$, and Newton's second law becomes:

$$\mathbf{F} = m\mathbf{a} = m\frac{d^2\mathbf{z}}{dt^2} = q\mathbf{E} = (-0.3 \times 10^{-6})(30\,\mathbf{a}_z)$$

describing motion of the charge in the z direction. The initial velocity in x is constant, and so no force is applied in that direction. We integrate once:

$$\frac{dz}{dt} = v_z = \frac{qE}{m}t + C_1$$

The initial velocity along z, $v_z(0)$ is zero, and so $C_1 = 0$. Integrating a second time yields the z coordinate:

$$z = \frac{qE}{2m}t^2 + C_2$$

The charge lies at the origin at $t = 0$, and so $C_2 = 0$. Introducing the given values, we find

$$z = \frac{(-0.3 \times 10^{-6})(30)}{2 \times 3 \times 10^{-16}}t^2 = -1.5 \times 10^{10} t^2 \text{ m}$$

At $t = 3\,\mu$s, $z = -(1.5 \times 10^{10})(3 \times 10^{-6})^2 = -.135$ cm. Now, considering the initial constant velocity in x, the charge in 3 μs attains an x coordinate of $x = vt = (3 \times 10^5)(3 \times 10^{-6}) = .90$ m. In summary, at $t = 3\,\mu$s we have $P(x, y, z) = \underline{(.90, 0, -.135)}$.

b) the velocity, $\mathbf{v}$: After the first integration in part a, we find

$$v_z = \frac{qE}{m}t = -(3 \times 10^{10})(3 \times 10^{-6}) = -9 \times 10^4 \text{ m/s}$$

Including the intial x-directed velocity, we finally obtain $\mathbf{v} = \underline{3 \times 10^5\,\mathbf{a}_x - 9 \times 10^4\mathbf{a}_z \text{ m/s}}$.

c) the kinetic energy of the charge: Have

$$\text{K.E.} = \frac{1}{2}m|v|^2 = \frac{1}{2}(3 \times 10^{-16})(1.13 \times 10^5)^2 = \underline{1.5 \times 10^{-5}\text{ J}}$$

9.2. A point charge, $Q = -0.3\,\mu$C and $m = 3 \times 10^{-16}$ kg, is moving through the field $\mathbf{B} = 30\mathbf{a}_z$ mT. Make use of Eq. (2) and Newton's laws to develop the appropriate differential equations, and solve them, subject to the initial condition at $t = 0$, $\mathbf{v} = 3 \times 10^5$ m/s at the origin. Solve these equations (perhaps with the help of an example given in Section 7.5) to evaluate at $t = 3\mu$s: a) the position $P(x, y, z)$ of the charge; b) its velocity; c) and its kinetic energy:

We begin by visualizing the problem. Using $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, we find that a *positive* charge moving along positive $\mathbf{a}_x$, would encounter the z-directed $\mathbf{B}$ field and be deflected into the *negative* y direction.

9.2 (continued) Motion along negative y through the field would cause further deflection into the *negative* x direction. We can construct the differential equations for the forces in x and in y as follows:

$$F_x \mathbf{a}_x = m \frac{dv_x}{dt} \mathbf{a}_x = q v_y \mathbf{a}_y \times B \mathbf{a}_z = q B v_y \mathbf{a}_x$$

$$F_y \mathbf{a}_y = m \frac{dv_y}{dt} \mathbf{a}_y = q v_x \mathbf{a}_x \times B \mathbf{a}_z = -q B v_x \mathbf{a}_y$$

or

$$\frac{dv_x}{dt} = \frac{qB}{m} v_y \tag{1}$$

and

$$\frac{dv_y}{dt} = -\frac{qB}{m} v_x \tag{2}$$

To solve these equations, we first differentiate (2) with time and substitute (1), obtaining:

$$\frac{d^2 v_y}{dt^2} = -\frac{qB}{m} \frac{dv_x}{dt} = -\left(\frac{qB}{m}\right)^2 v_y$$

Therefore, $v_y = A \sin(qBt/m) + A' \cos(qBt/m)$. However, at $t = 0$, $v_y = 0$, and so $A' = 0$, leaving $v_y = A \sin(qBt/m)$. Then, using (2),

$$v_x = -\frac{m}{qB} \frac{dv_y}{dt} = -A \cos\left(\frac{qBt}{m}\right)$$

Now at $t = 0$, $v_x = v_{x0} = 3 \times 10^5$. Therefore $A = -v_{x0}$, and so $v_x = v_{x0} \cos(qBt/m)$, and $v_y = -v_{x0} \sin(qBt/m)$. The positions are then found by integrating v_x and v_y over time:

$$x(t) = \int v_{x0} \cos\left(\frac{qBt}{m}\right) dt + C = \frac{m v_{x0}}{qB} \sin\left(\frac{qBt}{m}\right) + C$$

where $C = 0$, since $x(0) = 0$. Then

$$y(t) = \int -v_{x0} \sin\left(\frac{qBt}{m}\right) dt + D = \frac{m v_{x0}}{qB} \cos\left(\frac{qBt}{m}\right) + D$$

We require that $y(0) = 0$, so $D = -(m v_{x0})/(qB)$, and finally $y(t) = -m v_{x0}/qB \,[1 - \cos(qBt/m)]$. Summarizing, we have, using $q = -3 \times 10^{-7}$ C, $m = 3 \times 10^{-16}$ kg, $B = 30 \times 10^{-3}$ T, and $v_{x0} = 3 \times 10^5$ m/s:

$$x(t) = \frac{m v_{x0}}{qB} \sin\left(\frac{qBt}{m}\right) = -10^{-2} \sin(-3 \times 10^{-7} t) \text{ m}$$

$$y(t) = -\frac{m v_{x0}}{qB} \left[1 - \cos\left(\frac{qBt}{m}\right)\right] = 10^{-2}[1 - \cos(-3 \times 10^{7} t)] \text{ m}$$

$$v_x(t) = v_{x0} \cos\left(\frac{qBt}{m}\right) = 3 \times 10^5 \cos(-3 \times 10^7 t) \text{ m/s}$$

$$v_y(t) = -v_{x0} \sin\left(\frac{qBt}{m}\right) = -3 \times 10^5 \sin(-3 \times 10^7 t) \text{ m/s}$$

9.2 (continued) The answers are now:

a) At $t = 3 \times 10^{-6}$ s, $x = \underline{8.9 \text{ mm}}$, $y = \underline{14.5 \text{ mm}}$, and $z = \underline{0}$.

b) At $t = 3 \times 10^{-6}$ s, $v_x = -1.3 \times 10^5$ m/s, $v_y = 2.7 \times 10^5$ m/s, and so

$$\mathbf{v}(t = 3\,\mu\text{s}) = \underline{-1.3 \times 10^5 \mathbf{a}_x + 2.7 \times 10^5 \mathbf{a}_y \text{ m/s}}$$

whose magnitude is $v = 3 \times 10^5$ m/s as would be expected.

c) Kinetic energy is K.E. $= (1/2)mv^2 = \underline{1.35\ \mu\text{J}}$ at all times.

9.3. A point charge for which $Q = 2 \times 10^{-16}$ C and $m = 5 \times 10^{-26}$ kg is moving in the combined fields $\mathbf{E} = 100\mathbf{a}_x - 200\mathbf{a}_y + 300\mathbf{a}_z$ V/m and $\mathbf{B} = -3\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z$ mT. If the charge velocity at $t = 0$ is $\mathbf{v}(0) = (2\mathbf{a}_x - 3\mathbf{a}_y - 4\mathbf{a}_z) \times 10^5$ m/s:

a) give the unit vector showing the direction in which the charge is accelerating at $t = 0$: Use $\mathbf{F}(t = 0) = q[\mathbf{E} + (\mathbf{v}(0) \times \mathbf{B})]$, where

$$\mathbf{v}(0) \times \mathbf{B} = (2\mathbf{a}_x - 3\mathbf{a}_y - 4\mathbf{a}_z)10^5 \times (-3\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z)10^{-3} = 1100\mathbf{a}_x + 1400\mathbf{a}_y - 500\mathbf{a}_z$$

So the force in newtons becomes

$$\mathbf{F}(0) = (2\times10^{-16})[(100+1100)\mathbf{a}_x+(1400-200)\mathbf{a}_y+(300-500)\mathbf{a}_z] = 4\times10^{-14}[6\mathbf{a}_x+6\mathbf{a}_y-\mathbf{a}_z]$$

The unit vector that gives the acceleration direction is found from the force to be

$$\mathbf{a}_F = \frac{6\mathbf{a}_x + 6\mathbf{a}_y - \mathbf{a}_z}{\sqrt{73}} = \underline{.70\mathbf{a}_x + .70\mathbf{a}_y - .12\mathbf{a}_z}$$

b) find the kinetic energy of the charge at $t = 0$:

$$\text{K.E.} = \frac{1}{2}m|\mathbf{v}(0)|^2 = \frac{1}{2}(5 \times 10^{-26}\,\text{kg})(5.39 \times 10^5\,\text{m/s})^2 = 7.25 \times 10^{-15}\,\text{J} = \underline{7.25\text{ fJ}}$$

9.4. An electron ($q_e = -1.60219 \times 10^{-19}$ C, $m = 9.10956 \times 10^{-31}$ kg) is moving at a constant velocity $\mathbf{v} = 4.5 \times 10^7\mathbf{a_y}$ m/s along the negative y axis. At the origin it encounters the uniform magnetic field $\mathbf{B} = 2.5\mathbf{a}_z$ mT, and remains in it up to $y = 2.5$ cm. If we assume (with good accuracy) that the electron remains on the y axis while it is in the magnetic field, find its x-, y-, and z-coordinate values when $y = 50$ cm: The procedure is to find the electron velocity as it leaves the field, and then determine its coordinates at the time corresponding to $y = 50$ cm. The force it encounters while in the field is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = (-1.60219 \times 10^{-19})(4.5 \times 10^7)(2.5 \times 10^{-3})(\mathbf{a}_y \times \mathbf{a}_z) = -1.80 \times 10^{-14}\mathbf{a}_x\text{ N}$$

This force will be constant during the time the electron traverses the field. It establishes a negative x-directed velocity as it leaves the field, given by the acceleration times the transit time, t_t:

$$v_x = \frac{Ft_t}{m} = \left(\frac{-1.80 \times 10^{14}\,\text{N}}{9.10956 \times 10^{-31}\,\text{kg}}\right)\left(\frac{2.5 \times 10^{-2}\,\text{m}}{4.5 \times 10^7\,\text{m/s}}\right) = -1.09 \times 10^7\text{ m/s}$$

9.4 (continued) The time for the electron to travel along y between 2.5 and 50 cm is

$$t_{50} = \frac{(50 - 2.5) \times 10^{-2}}{4.5 \times 10^{7}} = 1.06 \times 10^{-8}\,\text{s}$$

In that time, the electron moves to an x coordinate given by

$$x = v_x t_{50} = -(1.09 \times 10^{7})(1.06 \times 10^{-8}) = -.115\,\text{m}$$

The coordinates at the time the electron reaches $y = 50$ cm are then:

$$\underline{x = -11.5\,\text{cm},\ \ y = 50\,\text{cm},\ \ z = 0}$$

9.5. A rectangular loop of wire in free space joins points $A(1, 0, 1)$ to $B(3, 0, 1)$ to $C(3, 0, 4)$ to $D(1, 0, 4)$ to A. The wire carries a current of 6 mA, flowing in the $\mathbf{a}_z$ direction from B to C. A filamentary current of 15 A flows along the entire z axis in the $\mathbf{a}_z$ direction.

a) Find $\mathbf{F}$ on side BC:

$$\mathbf{F}_{BC} = \int_B^C I_{\text{loop}} d\mathbf{L} \times \mathbf{B}_{\text{from wire at BC}}$$

Thus

$$\mathbf{F}_{BC} = \int_1^4 (6 \times 10^{-3})\, dz\, \mathbf{a}_z \times \frac{15\mu_0}{2\pi(3)}\, \mathbf{a}_y = -1.8 \times 10^{-8}\mathbf{a}_x \text{ N} = \underline{-18\mathbf{a}_x \text{ nN}}$$

b) Find $\mathbf{F}$ on side AB: The field from the long wire now varies with position along the loop segment. We include that dependence and write

$$\mathbf{F}_{AB} = \int_1^3 (6 \times 10^{-3})\, dx\, \mathbf{a}_x \times \frac{15\mu_0}{2\pi x}\, \mathbf{a}_y = \frac{45 \times 10^{-3}}{\pi}\mu_0 \ln 3\, \mathbf{a}_z = \underline{19.8\mathbf{a}_z \text{ nN}}$$

c) Find $\mathbf{F}_{\text{total}}$ on the loop: This will be the vector sum of the forces on the four sides. Note that by symmetry, the forces on sides AB and CD will be equal and opposite, and so will cancel. This leaves the sum of forces on sides BC (part a) and DA, where

$$\mathbf{F}_{DA} = \int_1^4 -(6 \times 10^{-3})\, dz\, \mathbf{a}_z \times \frac{15\mu_0}{2\pi(1)}\, \mathbf{a}_y = 54\mathbf{a}_x \text{ nN}$$

The total force is then $\mathbf{F}_{\text{total}} = \mathbf{F}_{DA} + \mathbf{F}_{BC} = (54 - 18)\mathbf{a}_x = \underline{36\,\mathbf{a}_x \text{ nN}}$

9.6 The magnetic flux density in a region of free space is given by $\mathbf{B} = -3x\mathbf{a}_x + 5y\mathbf{a}_y - 2z\mathbf{a}_z$ T. Find the total force on the rectangular loop shown in Fig. 9.15 if it lies in the plane $z = 0$ and is bounded by $x = 1$, $x = 3$, $y = 2$, and $y = 5$, all dimensions in cm: First, note that in the plane $z = 0$, the z component of the given field is zero, so will not contribute to the force. We use

$$\mathbf{F} = \int_{loop} I d\mathbf{L} \times \mathbf{B}$$

which in our case becomes, with $I = 30$ A:

$$\mathbf{F} = \int_{.01}^{.03} 30dx\mathbf{a}_x \times (-3x\mathbf{a}_x + 5y|_{y=.02}\, \mathbf{a}_y) + \int_{.02}^{.05} 30dy\mathbf{a}_y \times (-3x|_{x=.03}\, \mathbf{a}_x + 5y\mathbf{a}_y)$$
$$+ \int_{.03}^{.01} 30dx\mathbf{a}_x \times (-3x\mathbf{a}_x + 5y|_{y=.05}\, \mathbf{a}_y) + \int_{.05}^{.02} 30dy\mathbf{a}_y \times (-3x|_{x=.01}\, \mathbf{a}_x + 5y\mathbf{a}_y)$$

9.6. (continued) Simplifying, this becomes

$$\mathbf{F} = \int_{.01}^{.03} 30(5)(.02)\,\mathbf{a}_z\,dx + \int_{.02}^{.05} -30(3)(.03)(-\mathbf{a}_z)\,dy$$
$$+ \int_{.03}^{.01} 30(5)(.05)\,\mathbf{a}_z\,dx + \int_{.05}^{.02} -30(3)(.01)(-\mathbf{a}_z)\,dy = (.060 + .081 - .150 - .027)\mathbf{a}_z \text{ N}$$
$$= \underline{-36\,\mathbf{a}_z \text{ mN}}$$

9.7. Uniform current sheets are located in free space as follows: $8\mathbf{a}_z$ A/m at $y = 0$, $-4\mathbf{a}_z$ A/m at $y = 1$, and $-4\mathbf{a}_z$ A/m at $y = -1$. Find the vector force per meter length exerted on a current filament carrying 7 mA in the $\mathbf{a}_L$ direction if the filament is located at:

a) $x = 0$, $y = 0.5$, and $\mathbf{a}_L = \mathbf{a}_z$: We first note that within the region $-1 < y < 1$, the magnetic fields from the two outer sheets (carrying $-4\mathbf{a}_z$ A/m) cancel, leaving only the field from the center sheet. Therefore, $\mathbf{H} = -4\mathbf{a}_x$ A/m $(0 < y < 1)$ and $\mathbf{H} = 4\mathbf{a}_x$ A/m $(-1 < y < 0)$. Outside $(y > 1$ and $y < -1)$ the fields from all three sheets cancel, leaving $\mathbf{H} = 0$ $(y > 1, y < -1)$. So at $x = 0$, $y = .5$, the force per meter length will be

$$\mathbf{F}/\text{m} = I\mathbf{a}_z \times \mathbf{B} = (7 \times 10^{-3})\mathbf{a}_z \times -4\mu_0\mathbf{a}_x = \underline{-35.2\mathbf{a}_y \text{ nN/m}}$$

b.) $y = 0.5$, $z = 0$, and $\mathbf{a}_L = \mathbf{a}_x$: $\mathbf{F}/\text{m} = I\mathbf{a}_x \times -4\mu_0\mathbf{a}_x = \underline{0}$.

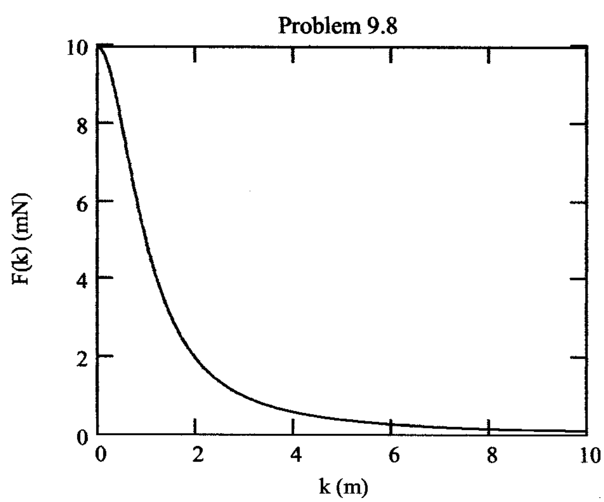
c) $x = 0$, $y = 1.5$, $\mathbf{a}_L = \mathbf{a}_z$: Since $y = 1.5$, we are in the region in which $\mathbf{B} = 0$, and so the force is zero.

9.8. Filamentary currents of $-25\mathbf{a}_z$ and $25\mathbf{a}_z$ A are located in the $x = 0$ plane in free space at $y = -1$ and $y = 1$m respectively. A third filamentary current of $10^{-3}\mathbf{a}_z$ A is located at $x = k$, $y = 0$. Find the vector force on a 1-m length of the 1-mA filament and plot $|\mathbf{F}|$ versus k: The total $\mathbf{B}$ field arising from the two 25A filaments evaluated at the location of the 1-mA filament is, in cartesian components:

$$\mathbf{B} = \underbrace{\frac{25\mu_0}{2\pi(1+k^2)}(k\mathbf{a}_y + \mathbf{a}_x)}_{\text{line at } y=+1} + \underbrace{\frac{25\mu_0}{2\pi(1+k^2)}(-k\mathbf{a}_y + \mathbf{a}_x)}_{\text{line at } y=-1} = \frac{25\mu_0\mathbf{a}_x}{\pi(1+k^2)}$$

The force on the 1m length of 1-mA line is now

$$\mathbf{F} = 10^{-3}(1)\mathbf{a}_z \times \frac{25\mu_0\mathbf{a}_x}{\pi(1+k^2)} = \frac{(2.5 \times 10^{-2})(4 \times 10^{-7})}{(1+k^2)}\mathbf{a}_y = \frac{10^{-8}\mathbf{a}_y}{(1+k^2)}\mathbf{a}_y \text{ N} = \underline{\frac{10\mathbf{a}_y}{(1+k^2)} \text{ nN}}$$



9.9. A current of $-100\mathbf{a}_z$ A/m flows on the conducting cylinder $\rho = 5$ mm and $+500\mathbf{a}_z$ A/m is present on the conducting cylinder $\rho = 1$ mm. Find the magnitude of the total force acting to split the outer cylinder apart along its length: The differential force acting on the outer cylinder arising from the field of the inner cylinder is $d\mathbf{F} = \mathbf{K}_{\text{outer}} \times \mathbf{B}$, where $\mathbf{B}$ is the field from the inner cylinder, evaluated at the outer cylinder location:

$$\mathbf{B} = \frac{2\pi(1)(500)\mu_0}{2\pi(5)}\mathbf{a}_\phi = 100\mu_0\,\mathbf{a}_\phi \text{ T}$$

Thus $d\mathbf{F} = -100\mathbf{a}_z \times 100\mu_0\mathbf{a}_\phi = 10^4\mu_0\mathbf{a}_\rho$ N/m^2. We wish to find the force acting to split the outer cylinder, which means we need to evaluate the net force in one cartesian direction on one half of the cylinder. We choose the "upper" half $(0 < \phi < \pi)$, and integrate the y component of $d\mathbf{F}$ over this range, and over a unit length in the z direction:

$$F_y = \int_0^1 \int_0^\pi 10^4\mu_0\mathbf{a}_\rho \cdot \mathbf{a}_y(5 \times 10^{-3})\,d\phi\,dz = \int_0^\pi 50\mu_0 \sin\phi\,d\phi = 100\mu_0 = \underline{4\pi \times 10^{-5} \text{ N/m}}$$

Note that we did not include the "self force" arising from the outer cylinder's $\mathbf{B}$ field on itself. Since the outer cylinder is a two-dimensional current sheet, its field exists only just outside the cylinder, and so no force exists. If this cylinder possessed a finite thickness, then we would need to include its self-force, since there would be an interior field and a volume current density that would spatially overlap.

9.10. Two infinitely-long parallel filaments each carry 50 A in the $\mathbf{a}_z$ direction. If the filaments lie in the plane $y = 0$ at $x = 0$ and $x = 5$mm (note bad wording in problem statement in book), find the vector force per meter length on the filament passing through the origin: The force will be

$$\mathbf{F} = \int_0^1 I\,d\mathbf{L} \times \mathbf{B}$$

where $I\,d\mathbf{L}$ is that of the filament at the origin, and $\mathbf{B}$ is that arising from the filament at $x = 5$mm evaluated at the location of the other filament (along the z axis). We obtain

$$\mathbf{F} = \int_0^1 50\,dz\mathbf{a}_z \times \frac{-50\mu_0\mathbf{a}_y}{2\pi(5 \times 10^{-3})} = \underline{0.10\,\mathbf{a}_x \text{ N/m}}$$

9.11. a) Use Eq. (14), Sec. 9.3, to show that the force of attraction per unit length between two filamentary conductors in free space with currents $I_1\mathbf{a}_z$ at $x = 0$, $y = d/2$, and $I_2\mathbf{a}_z$ at $x = 0$, $y = -d/2$, is $\mu_0 I_1 I_2/(2\pi d)$: The force on I_2 is given by

$$\mathbf{F}_2 = \mu_0\frac{I_1 I_2}{4\pi}\oint\left[\oint \frac{\mathbf{a}_{R12} \times d\mathbf{L}_1}{R_{12}^2}\right] \times d\mathbf{L}_2$$

Let z_1 indicate the z coordinate along I_1, and z_2 indicate the z coordinate along I_2. We then have $R_{12} = \sqrt{(z_2 - z_1)^2 + d^2}$ and

$$\mathbf{a}_{R12} = \frac{(z_2 - z_1)\mathbf{a}_z - d\mathbf{a}_y}{\sqrt{(z_2 - z_1)^2 + d^2}}$$

Also, $d\mathbf{L}_1 = dz_1\mathbf{a}_z$ and $d\mathbf{L}_2 = dz_2\mathbf{a}_z$ The "inside" integral becomes:

$$\oint \frac{\mathbf{a}_{R12} \times d\mathbf{L}_1}{R_{12}^2} = \oint \frac{[(z_2 - z_1)\mathbf{a}_z - d\mathbf{a}_y] \times dz_1\mathbf{a}_z}{[(z_2 - z_1)^2 + d^2]^{1.5}} = \int_{-\infty}^{\infty} \frac{-d\,dz_1\,\mathbf{a}_x}{[(z_2 - z_1)^2 + d^2]^{1.5}}$$

9.11a. (continued) The force expression now becomes

$$\mathbf{F}_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[\int_{-\infty}^{\infty} \frac{-d\, dz_1\, \mathbf{a}_x}{[(z_2 - z_1)^2 + d^2]^{1.5}} \times dz_2 \mathbf{a}_z \right] = \mu_0 \frac{I_1 I_2}{4\pi} \int_0^1 \int_{-\infty}^{\infty} \frac{d\, dz_1\, dz_2\, \mathbf{a}_y}{[(z_2 - z_1)^2 + d^2]^{1.5}}$$

Note that the "outside" integral is taken over a unit length of current I_2. Evaluating, obtain,

$$\mathbf{F}_2 = \mu_0 \frac{I_1 I_2 d\, \mathbf{a}_y}{4\pi d^2} (2) \int_0^1 dz_2 = \frac{\mu_0 I_1 I_2}{2\pi d} \mathbf{a}_y \text{ N/m}$$

as expected.

b) Show how a simpler method can be used to check your result: We use $d\mathbf{F}_2 = I_2 d\mathbf{L}_2 \times \mathbf{B}_{12}$, where the field from current 1 at the location of current 2 is

$$\mathbf{B}_{12} = \frac{\mu_0 I_1}{2\pi d} \mathbf{a}_x \text{ T}$$

so over a unit length of I_2, we obtain

$$\mathbf{F}_2 = I_2 \mathbf{a}_z \times \frac{\mu_0 I_1}{2\pi d} \mathbf{a}_x = \mu_0 \frac{I_1 I_2}{2\pi d} \mathbf{a}_y \text{ N/m}$$

This second method is really just the first over again, since we recognize the inside integral of the first method as the Biot-Savart law, used to find the field from current 1 at the current 2 location.

9.12. A conducting current strip carrying $\mathbf{K} = 12\mathbf{a}_z$ A/m lies in the $x = 0$ plane between $y = 0.5$ and $y = 1.5$ m. There is also a current filament of $I = 5$ A in the $\mathbf{a}_z$ direction on the z axis. Find the force exerted on the:

a) filament by the current strip: We first need to find the field from the current strip at the filament location. Consider the strip as made up of many adjacent strips of width dy, each carrying current $dI\mathbf{a}_z = \mathbf{K}dy$. The field along the z axis from each differential strip will be $d\mathbf{B} = [(K dy \mu_0)/(2\pi y)]\mathbf{a}_x$. The total $\mathbf{B}$ field from the strip evaluated along the z axis is therefore

$$\mathbf{B} = \int_{0.5}^{1.5} \frac{12\mu_0 \mathbf{a}_x}{2\pi y} dy = \frac{6\mu_0}{\pi} \ln\left(\frac{1.5}{0.5}\right) \mathbf{a}_x = 2.64 \times 10^{-6} \mathbf{a}_x \text{ Wb/m}^2$$

Now

$$\mathbf{F} = \int_0^1 I d\mathbf{L} \times \mathbf{B} = \int_0^1 5dz\, \mathbf{a}_z \times 2.64 \times 10^{-6}\, \mathbf{a}_x\, dz = \underline{13.2\, \mathbf{a}_y\ \mu\text{N/m}}$$

b) strip by the filament: In this case we integrate $\mathbf{K} \times \mathbf{B}$ over a unit length in z of the strip area, where $\mathbf{B}$ is the field from the filament evaluated on the strip surface:

$$\mathbf{F} = \int_{Area} \mathbf{K} \times \mathbf{B}\, da = \int_0^1 \int_{0.5}^{1.5} 12\mathbf{a}_z \times \frac{-5\mu_0 \mathbf{a}_x}{2\pi y} dy = \frac{-30\mu_0}{\pi} \ln(3)\, \mathbf{a}_y = \underline{-13.2\, \mathbf{a}_y\ \mu\text{N/m}}$$

9.13. A current of 6A flows from $M(2,0,5)$ to $N(5,0,5)$ in a straight solid conductor in free space. An infinite current filament lies along the z axis and carries 50A in the $\mathbf{a}_z$ direction. Compute the vector torque on the wire segment using:

a) an origin at $(0,0,5)$: The $\mathbf{B}$ field from the long wire at the short wire is $\mathbf{B} = (\mu_0 I_z \mathbf{a_y})/(2\pi x)$ T. Then the force acting on a differential length of the wire segment is

$$d\mathbf{F} = I_w d\mathbf{L} \times \mathbf{B} = I_w dx\, \mathbf{a}_x \times \frac{\mu_0 I_z}{2\pi x}\mathbf{a}_y = \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z \text{ N}$$

Now the differential torque about $(0,0,5)$ will be

$$d\mathbf{T} = \mathbf{R}_T \times d\mathbf{F} = x\mathbf{a}_x \times \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z = -\frac{\mu_0 I_w I_z}{2\pi}\, dx\, \mathbf{a}_y$$

The net torque is now found by integrating the differential torque over the length of the wire segment:

$$\mathbf{T} = \int_2^5 -\frac{\mu_0 I_w I_z}{2\pi}\, dx\, \mathbf{a}_y = -\frac{3\mu_0(6)(50)}{2\pi}\, \mathbf{a}_y = \underline{-1.8 \times 10^{-4}\, \mathbf{a}_y \ \text{ N}\cdot\text{m}}$$

b) an origin at $(0,0,0)$: Here, the only modification is in $\mathbf{R}_T$, which is now $\mathbf{R}_T = x\,\mathbf{a}_x + 5\,\mathbf{a}_z$ So now

$$d\mathbf{T} = \mathbf{R}_T \times d\mathbf{F} = \left[x\mathbf{a}_x + 5\mathbf{a}_z\right] \times \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z = -\frac{\mu_0 I_w I_z}{2\pi}\, dx\, \mathbf{a}_y$$

Everything from here is the same as in part a, so again, $\mathbf{T} = \underline{-1.8 \times 10^{-4}\, \mathbf{a}_y \ \text{ N}\cdot\text{m}}$.

c) an origin at $(3,0,0)$: In this case, $\mathbf{R}_T = (x-3)\mathbf{a}_x + 5\mathbf{a}_z$, and the differential torque is

$$d\mathbf{T} = \left[(x-3)\mathbf{a}_x + 5\mathbf{a}_z\right] \times \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z = -\frac{\mu_0 I_w I_z (x-3)}{2\pi x}\, dx\, \mathbf{a}_y$$

Thus

$$\mathbf{T} = \int_2^5 -\frac{\mu_0 I_w I_z (x-3)}{2\pi x}\, dx\, \mathbf{a}_y = -6.0 \times 10^{-5}\left[3 - 3\ln\left(\frac{5}{2}\right)\right]\mathbf{a}_y = \underline{-1.5 \times 10^{-5}\, \mathbf{a}_y \ \text{ N}\cdot\text{m}}$$

9.14. The rectangular loop of Prob. 6 is now subjected to the $\mathbf{B}$ field produced by two current sheets, $\mathbf{K}_1 = 400\,\mathbf{a}_y$ A/m at $z = 2$, and $\mathbf{K}_2 = 300\,\mathbf{a}_z$ A/m at $y = 0$ in free space. Find the vector torque on the loop, referred to an origin:

a) at (0,0,0): The fields from both current sheets, at the loop location, will be negative x-directed. They will add together to give, in the loop plane:

$$\mathbf{B} = -\mu_0\left(\frac{K_1}{2} + \frac{K_2}{2}\right)\mathbf{a}_x = -\mu_0(200 + 150)\,\mathbf{a}_x = -350\mu_0\,\mathbf{a}_x \text{ Wb/m}^2$$

With this field, forces will be acting only on the wire segments that are parallel to the y axis. The force on the segment nearer to the y axis will be

$$\mathbf{F}_1 = I\mathbf{L} \times \mathbf{B} = -30(3 \times 10^{-2})\mathbf{a}_y \times -350\mu_0\mathbf{a}_x = -315\mu_0\,\mathbf{a}_z \text{ N}$$

9.14a (continued) The force acting on the segment farther from the y axis will be

$$\mathbf{F}_2 = I\mathbf{L} \times \mathbf{B} = 30(3 \times 10^{-2})\mathbf{a}_y \times -350\mu_0 \mathbf{a}_x = 315\mu_0\, \mathbf{a}_z \text{ N}$$

The torque about the origin is now $\mathbf{T} = \mathbf{R}_1 \times \mathbf{F}_1 + \mathbf{R}_2 \times \mathbf{F}_2$, where $\mathbf{R}_1$ is the vector directed from the origin to the midpoint of the nearer y-directed segment, and $\mathbf{R}_2$ is the vector joining the origin to the midpoint of the farther y-directed segment. So $\mathbf{R}_1(\text{cm}) = \mathbf{a}_x + 3.5\mathbf{a}_y$ and $\mathbf{R}_2(\text{cm}) = 3\mathbf{a}_x + 3.5\mathbf{a}_y$. Therefore

$$\begin{aligned}\mathbf{T}_{0,0,0} &= [(\mathbf{a}_x + 3.5\mathbf{a}_y) \times 10^{-2}] \times -315\mu_0\, \mathbf{a}_z + [(3\mathbf{a}_x + 3.5\mathbf{a}_y) \times 10^{-2}] \times 315\mu_0\, \mathbf{a}_z \\ &= -6.30\mu_0 \mathbf{a}_y = \underline{-7.92 \times 10^{-6}\, \mathbf{a}_y \text{ N}-\text{m}}\end{aligned}$$

b) at the center of the loop: Use $\mathbf{T} = I\mathbf{S} \times \mathbf{B}$ where $\mathbf{S} = (2 \times 3) \times 10^{-4}\, \mathbf{a}_z \text{ m}^2$. So

$$\mathbf{T} = 30(6 \times 10^{-4}\mathbf{a}_z) \times (-350\mu_0\, \mathbf{a}_x) = \underline{-7.92 \times 10^{-6}\, \mathbf{a}_y \text{ N}-\text{m}}$$

9.15. A solid conducting filament extends from $x = -b$ to $x = b$ along the line $y = 2$, $z = 0$. This filament carries a current of 3 A in the $\mathbf{a}_x$ direction. An infinite filament on the z axis carries 5 A in the $\mathbf{a}_z$ direction. Obtain an expression for the torque exerted on the finite conductor about an origin located at $(0, 2, 0)$: The differential force on the wire segment arising from the field from the infinite wire is

$$d\mathbf{F} = 3\,dx\,\mathbf{a}_x \times \frac{5\mu_0}{2\pi\rho}\,\mathbf{a}_\phi = -\frac{15\mu_0 \cos\phi\, dx}{2\pi\sqrt{x^2+4}}\,\mathbf{a}_z = -\frac{15\mu_0 x\, dx}{2\pi(x^2+4)}\,\mathbf{a}_z$$

So now the differential torque about the $(0, 2, 0)$ origin is

$$d\mathbf{T} = \mathbf{R}_T \times d\mathbf{F} = x\,\mathbf{a}_x \times -\frac{15\mu_0 x\, dx}{2\pi(x^2+4)}\,\mathbf{a}_z = \frac{15\mu_0 x^2\, dx}{2\pi(x^2+4)}\,\mathbf{a}_y$$

The torque is then

$$\begin{aligned}\mathbf{T} &= \int_{-b}^{b} \frac{15\mu_0 x^2\, dx}{2\pi(x^2+4)}\,\mathbf{a}_y = \frac{15\mu_0}{2\pi}\,\mathbf{a}_y \left[x - 2\tan^{-1}\left(\frac{x}{2}\right)\right]_{-b}^{b} \\ &= \underline{(6 \times 10^{-6})\left[b - 2\tan^{-1}\left(\frac{b}{2}\right)\right]\mathbf{a}_y \ \text{N} \cdot \text{m}}\end{aligned}$$

9.16. Assume that an electron is describing a circular orbit of radius a about a positively-charged nucleus.

a) By selecting an appropriate current and area, show that the equivalent orbital dipole moment is $ea^2\omega/2$, where ω is the electron's angular velocity: The current magnitude will be $I = \frac{e}{T}$, where e is the electron charge and T is the orbital period. The latter is $T = 2\pi/\omega$, and so $I = e\omega/(2\pi)$. Now the dipole moment magnitude will be $m = IA$, where A is the loop area. Thus

$$m = \frac{e\omega}{2\pi}\,\pi a^2 = \frac{1}{2}ea^2\omega \ \ //$$

b) Show that the torque produced by a magnetic field parallel to the plane of the orbit is $ea^2\omega B/2$: With B assumed constant over the loop area, we would have $\mathbf{T} = \mathbf{m} \times \mathbf{B}$. With $\mathbf{B}$ parallel to the loop plane, $\mathbf{m}$ and $\mathbf{B}$ are orthogonal, and so $T = mB$. So, using part a, $T = ea^2\omega B/2$.

9.16. (continued)

c) by equating the Coulomb and centrifugal forces, show that ω is $(4\pi\epsilon_0 m_e a^3/e^2)^{-1/2}$, where m_e is the electron mass: The force balance is written as

$$\frac{e^2}{4\pi\epsilon_0 a^2} = m^e \omega^2 a \;\;\Rightarrow\;\; \omega = \left(\frac{4\pi\epsilon_0 m_e a^3}{e^2}\right)^{-1/2} \quad //$$

d) Find values for the angular velocity, torque, and the orbital magnetic moment for a hydrogen atom, where a is about 6×10^{-11} m; let $B = 0.5$ T: First

$$\omega = \left[\frac{(1.60 \times 10^{-19})^2}{4\pi(8.85 \times 10^{-12})(9.1 \times 10^{-31})(6 \times 10^{-11})^3}\right]^{1/2} = \underline{3.42 \times 10^{16}\ \text{rad/s}}$$

$$T = \frac{1}{2}(3.42 \times 10^{16})(1.60 \times 10^{-19})(0.5)(6 \times 10^{-11})^2 = \underline{4.93 \times 10^{-24}\ \text{N} \cdot \text{m}}$$

Finally,

$$m = \frac{T}{B} = \underline{9.86 \times 10^{-24}\ \text{A} \cdot \text{m}^2}$$

9.17. The hydrogen atom described in Problem 16 is now subjected to a magnetic field having the same direction as that of the atom. Show that the forces caused by B result in a decrease of the angular velocity by $eB/(2m_e)$ and a decrease in the orbital moment by $e^2a^2B/(4m_e)$. What are these decreases for the hydrogen atom in parts per million for an external magnetic flux density of 0.5 T? We first write down all forces on the electron, in which we equate its coulomb force toward the nucleus to the sum of the centrifugal force and the force associated with the applied B field. With the field applied in the same direction as that of the atom, this would yield a Lorentz force that is radially outward – in the same direction as the centrifugal force.

$$F_e = F_{cent} + F_B \;\;\Rightarrow\;\; \frac{e^2}{4\pi\epsilon_0 a^2} = m_e \omega^2 a + \underbrace{e\omega a B}_{QvB}$$

With $B = 0$, we solve for ω to find:

$$\omega = \omega_0 = \sqrt{\frac{e^2}{4\pi\epsilon_0 m_e a^3}}$$

Then with B present, we find

$$\omega^2 = \frac{e^2}{4\pi\epsilon_0 m_e a^3} - \frac{e\omega B}{m_e} = \omega_0^2 - \frac{e\omega B}{m_e}$$

Therefore

$$\omega = \omega_0 \sqrt{1 - \frac{e\omega B}{\omega_0^2 m_e}} \doteq \omega_0 \left(1 - \frac{e\omega B}{2\omega_0^2 m_e}\right)$$

But $\omega \doteq \omega_0$, and so

$$\omega \doteq \omega_0 \left(1 - \frac{eB}{2\omega_0 m_e}\right) = \omega_0 - \frac{eB}{2m_e} \quad //$$

9.17. (continued) As for the magnetic moment, we have

$$m = IS = \frac{e\omega}{2\pi}\pi a^2 = \frac{1}{2}\omega e a^2 \doteq \frac{1}{2}ea^2\left(\omega_0 - \frac{eB}{2m_e}\right) = \frac{1}{2}\omega_0 e a^2 - \frac{1}{4}\frac{e^2a^2B}{m_e} \quad //$$

Finally, for $a = 6 \times 10^{-11}$ m, $B = 0.5$ T, we have

$$\frac{\Delta\omega}{\omega} = \frac{eB}{2m_e}\frac{1}{\omega} \doteq \frac{eB}{2m_e}\frac{1}{\omega_0} = \frac{1.60 \times 10^{-19} \times 0.5}{2 \times 9.1 \times 10^{-31} \times 3.4 \times 10^{16}} = \underline{1.3 \times 10^{-6}}$$

where $\omega_0 = 3.4 \times 10^{16}$ sec^{-1} is found from Problem 16. Finally,

$$\frac{\Delta m}{m} = \frac{e^2a^2B}{4m_e} \times \frac{2}{\omega e a^2} \doteq \frac{eB}{2m_e\omega_0} = \underline{1.3 \times 10^{-6}}$$

9.18. Calculate the vector torque on the square loop shown in Fig. 9.16 about an origin at A in the field $\mathbf{B}$, given:

a) $A(0, 0, 0)$ and $\mathbf{B} = 100\mathbf{a}_y$ mT: The field is uniform and so does not produce any translation of the loop. Therefore, we may use $\mathbf{T} = I\mathbf{S} \times \mathbf{B}$ about any origin, where $I = 0.6$ A and $\mathbf{S} = 16\mathbf{a}_z$ m^2. We find $\mathbf{T} = 0.6(16)\mathbf{a}_z \times 0.100\mathbf{a}_y = \underline{-0.96\,\mathbf{a}_x \text{ N}-\text{m}}$.

b) $A(0, 0, 0)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y$ mT: Using the same reasoning as in part a, we find

$$\mathbf{T} = 0.6(16)\mathbf{a}_z \times (0.200\mathbf{a}_x + 0.100\mathbf{a}_y) = \underline{-0.96\mathbf{a}_x + 1.92\mathbf{a}_y \text{ N}-\text{m}}$$

c) $A(1, 2, 3)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y - 300\mathbf{a}_z$ mT: We observe two things here: 1) The field is again uniform and so again the torque is independent of the origin chosen, and 2) The field differs from that of part b only by the addition of a z component. With $\mathbf{S}$ in the z direction, this new component of $\mathbf{B}$ will produce no torque, so the answer is the *same as part b*, or $\mathbf{T} = \underline{-0.96\mathbf{a}_x + 1.92\mathbf{a}_y \text{ N}-\text{m}}$.

d) $A(1, 2, 3)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y - 300\mathbf{a}_z$ mT for $x \geq 2$ and $\mathbf{B} = 0$ elsewhere: Now, force is acting only on the y-directed segment at $x = +2$, so we need to be careful, since translation will occur. So we must use the given origin. The differential torque acting on the differential wire segment at location (2,y) is $d\mathbf{T} = \mathbf{R}(y) \times d\mathbf{F}$, where

$$d\mathbf{F} = Id\mathbf{L} \times \mathbf{B} = 0.6\,dy\,\mathbf{a}_y \times [0.2\mathbf{a}_x + 0.1\mathbf{a}_y - 0.3\mathbf{a}_z] = [-0.18\mathbf{a}_x - 0.12\mathbf{a}_z]\,dy$$

and $\mathbf{R}(y) = (2, y, 0) - (1, 2, 3) = \mathbf{a}_x + (y-2)\mathbf{a}_y - 3\mathbf{a}_z$. We thus find

$$\begin{aligned} d\mathbf{T} = \mathbf{R}(y) \times d\mathbf{F} &= \left[\mathbf{a}_x + (y-2)\mathbf{a}_y - 3\mathbf{a}_z\right] \times [-0.18\mathbf{a}_x - 0.12\mathbf{a}_z]\,dy \\ &= \left[-0.12(y-2)\mathbf{a}_x + 0.66\mathbf{a}_y + 0.18(y-2)\mathbf{a}_z\right]\,dy \end{aligned}$$

The net torque is now

$$\mathbf{T} = \int_{-2}^{2}\left[-0.12(y-2)\mathbf{a}_x + 0.66\mathbf{a}_y + 0.18(y-2)\mathbf{a}_z\right]\,dy = \underline{0.96\mathbf{a}_x + 2.64\mathbf{a}_y - 1.44\mathbf{a}_z \text{ N}-\text{m}}$$

9.19. Given a material for which $\chi_m = 3.1$ and within which $\mathbf{B} = 0.4y\mathbf{a}_z$ T, find:

a) $\mathbf{H}$: We use $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H}$, or

$$\mathbf{H} = \frac{0.4y\mathbf{a}_y}{(1+3.1)\mu_0} = \underline{77.6y\mathbf{a}_z \text{ kA/m}}$$

b) $\mu = (1+3.1)\mu_0 = \underline{5.15 \times 10^{-6} \text{ H/m}}$.

c) $\mu_R = (1+3.1) = \underline{4.1}$.

d) $\mathbf{M} = \chi_m \mathbf{H} = (3.1)(77.6y\mathbf{a}_y) = \underline{241y\mathbf{a}_z \text{ kA/m}}$

e) $\mathbf{J} = \nabla \times \mathbf{H} = (dH_z)/(dy)\,\mathbf{a}_x = \underline{77.6\,\mathbf{a}_x \text{ kA/m}^2}$.

f) $\mathbf{J}_b = \nabla \times \mathbf{M} = (dM_z)/(dy)\,\mathbf{a}_x = \underline{241\,\mathbf{a}_x \text{ kA/m}^2}$.

g) $\mathbf{J}_T = \nabla \times \mathbf{B}/\mu_0 = \underline{318\mathbf{a}_x \text{ kA/m}^2}$.

9.20. Find $\mathbf{H}$ in a material where:

a) $\mu_R = 4.2$, there are 2.7×10^{29} atoms/m^3, and each atom has a dipole moment of $2.6 \times 10^{-30}\,\mathbf{a}_y$ A $\cdot$ m^2. Since all dipoles are identical, we may write $\mathbf{M} = N\mathbf{m} = (2.7 \times 10^{29})(2.6 \times 10^{-30}\mathbf{a}_y) = 0.70\mathbf{a}_y$ A/m. Then

$$\mathbf{H} = \frac{\mathbf{M}}{\mu_R - 1} = \frac{0.70\,\mathbf{a}_y}{4.2 - 1} = \underline{0.22\,\mathbf{a}_y \text{ A/m}}$$

b) $\mathbf{M} = 270\,\mathbf{a}_z$ A/m and $\mu = 2\ \mu$H/m: Have $\mu_R = \mu/\mu_0 = (2 \times 10^{-6})/(4\pi \times 10^{-7}) = 1.59$. Then $\mathbf{H} = 270\mathbf{a}_z/(1.59 - 1) = \underline{456\,\mathbf{a}_z \text{ A/m}}$.

c) $\chi_m = 0.7$ and $\mathbf{B} = 2\mathbf{a}_z$ T: Use

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0(1+\chi_m)} = \frac{2\mathbf{a}_z}{(4\pi \times 10^{-7})(1.7)} = \underline{936\,\mathbf{a}_z \text{ kA/m}}$$

d) Find $\mathbf{M}$ in a material where bound surface current densities of $12\,\mathbf{a}_z$ A/m and $-9\,\mathbf{a}_z$ A/m exist at $\rho = 0.3$ m and $\rho = 0.4$ m, respectively: We use $\oint \mathbf{M} \cdot d\mathbf{L} = I_b$, where, since currents are in the z direction and are symmetric about the z axis, we chose the path integrals to be circular loops centered on and normal to z. From the symmetry, $\mathbf{M}$ will be ϕ-directed and will vary only with radius. Note first that for $\rho < 0.3$ m, no bound current will be enclosed by a path integral, so we conclude that $\mathbf{M} = \underline{0 \text{ for } \rho < 0.3\text{m}}$. At radii between the currents the path integral will enclose only the inner current so,

$$\oint \mathbf{M} \cdot d\mathbf{L} = 2\pi\rho M_\phi = 2\pi(0.3)12 \;\Rightarrow\; \mathbf{M} = \underline{\frac{3.6}{\rho}\,\mathbf{a}_\phi \text{ A/m} \;\; (0.3 < \rho < 0.4\text{m})}$$

Finally, for $\rho > 0.4$ m, the total enclosed bound current is $I_{b,tot} = 2\pi(0.3)(12) - 2\pi(0.4)(9) = 0$, so therefore $\mathbf{M} = \underline{0 \ (\rho > 0.4\text{m})}$.

9.21. Find the magnitude of the magnetization in a material for which:

a) the magnetic flux density is 0.02 Wb/m^2 *and the magnetic susceptibility is 0.003* (note that this latter quantity is missing in the original problem statement): From $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ and from $\mathbf{M} = \chi_m \mathbf{H}$, we write

$$M = \frac{B}{\mu_0}\left(\frac{1}{\chi_m} + 1\right)^{-1} = \frac{B}{\mu_0(334)} = \frac{0.02}{(4\pi \times 10^{-7})(334)} = \underline{47.7 \text{ A/m}}$$

9.21b) the magnetic field intensity is 1200 A/m and the relative permeability is 1.005: From $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0\mu_R\mathbf{H}$, we write

$$M = (\mu_R - 1)H = (.005)(1200) = \underline{6.0\ \text{A/m}}$$

c) there are 7.2×10^{28} atoms per cubic meter, each having a dipole moment of $4 \times 10^{-30}\ \text{A}\cdot\text{m}^2$ in the same direction, and the magnetic susceptibility is 0.0003: With all dipoles identical the dipole moment density becomes

$$M = n\,m = (7.2 \times 10^{28})(4 \times 10^{-30}) = \underline{0.288\ \text{A/m}}$$

9.22. Three current sheets are located as follows: $160\mathbf{a}_z$ A/m at $x = 1$cm, $-40\mathbf{a}_z$ A/m at $x = 5$cm, and $50\mathbf{a}_z$ A/m at $x = 8$cm. Let $\mu = \mu_0$ for $x < 1$cm and $x > 8$cm; for $1 < x < 5$ cm, $\mu = 3\mu_0$, and for $5 < x < 8$cm, $\mu = 2\mu_0$. Find $\mathbf{B}$ everywhere: We know that the $\mathbf{H}$ field from an infinite current sheet will be given in magnitude by $H = K/2$, and will be directed parallel to the sheet and perpendicular to the current, with the directions on either side of the sheet determined by the right hand rule. With this in mind, we can construct the following expressions for the $\mathbf{B}$ field in all four regions:

$$\mathbf{B}(x < 1) = \frac{1}{2}\mu_0(-160 + 40 - 50) = \underline{-1.07 \times 10^{-4}\,\mathbf{a}_y\ \text{T}}$$

$$\mathbf{B}(1 < x < 5) = \frac{1}{2}(3\mu_0)(160 + 40 - 50) = \underline{2.83 \times 10^{-4}\,\mathbf{a}_y\ \text{T}}$$

$$\mathbf{B}(5 < x < 8) = \frac{1}{2}(2\mu_0)(160 - 40 - 50) = \underline{8.80 \times 10^{-5}\,\mathbf{a}_y\ \text{T}}$$

$$\mathbf{B}(x > 8) = \frac{1}{2}\mu_0(160 - 40 + 50) = \underline{1.07 \times 10^{-4}\,\mathbf{a}_y\ \text{T}}$$

9.23. Calculate values for H_ϕ, B_ϕ, and M_ϕ at $\rho = c$ for a coaxial cable with $a = 2.5$ mm and $b = 6$ mm if it carries current $I = 12$ A in the center conductor, and $\mu = 3\ \mu$H/m for $2.5 < \rho < 3.5$ mm, $\mu = 5\ \mu$H/m for $3.5 < \rho < 4.5$ mm, and $\mu = 10\ \mu$H/m for $4.5 < \rho < 6$ mm. Compute for:

a) $c = 3$ mm: Have

$$H_\phi = \frac{I}{2\pi\rho} = \frac{12}{2\pi(3 \times 10^{-3})} = \underline{637\ \text{A/m}}$$

Then $B_\phi = \mu H_\phi = (3 \times 10^{-6})(637) = \underline{1.91 \times 10^{-3}\ \text{Wb/m}^2}$.

Finally, $M_\phi = (1/\mu_0)B_\phi - H_\phi = \underline{884\ \text{A/m}}$.

b) $c = 4$ mm: Have

$$H_\phi = \frac{I}{2\pi\rho} = \frac{12}{2\pi(4 \times 10^{-3})} = \underline{478\ \text{A/m}}$$

Then $B_\phi = \mu H_\phi = (5 \times 10^{-6})(478) = \underline{2.39 \times 10^{-3}\ \text{Wb/m}^2}$.

Finally, $M_\phi = (1/\mu_0)B_\phi - H_\phi = \underline{1.42 \times 10^3\ \text{A/m}}$.

c) $c = 5$ mm: Have

$$H_\phi = \frac{I}{2\pi\rho} = \frac{12}{2\pi(5 \times 10^{-3})} = \underline{382\ \text{A/m}}$$

Then $B_\phi = \mu H_\phi = (10 \times 10^{-6})(382) = \underline{3.82 \times 10^{-3}\ \text{Wb/m}^2}$.

Finally, $M_\phi = (1/\mu_0)B_\phi - H_\phi = \underline{2.66 \times 10^3\ \text{A/m}}$.

9.24. A coaxial transmission line has $a = 5$ mm and $b = 20$ mm. Let its center lie on the z axis and let a dc current I flow in the $\mathbf{a}_z$ direction in the center conductor. The volume between the conductors contains a magnetic material for which $\mu_R = 2.5$, as well as air. Find $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{M}$ everywhere between conductors if $H_\phi = 600/\pi$ A/m at $\rho = 10$ mm, $\phi = \pi/2$, and the magnetic material is located where:

a) $a < \rho < 3a$; First, we know that $H_\phi = I/2\pi\rho$, from which we construct:

$$\frac{I}{2\pi(10^{-2})} = \frac{600}{\pi} \quad \Rightarrow \quad I = 12\,\text{A}$$

Since the interface between the two media lies in the $\mathbf{a}_\phi$ direction, we use the boundary condition of continuity of tangential $\mathbf{H}$ and write

$$\mathbf{H}(5 < \rho < 20) = \frac{12}{2\pi\rho}\mathbf{a}_\phi = \underline{\frac{6}{\pi\rho}\mathbf{a}_\phi \text{ A/m}}$$

In the magnetic material, we find

$$\mathbf{B}(5 < \rho < 15) = \mu\mathbf{H} = \frac{(2.5)(4\pi \times 10^{-7})(12)}{2\pi\rho}\mathbf{a}_\phi = \underline{(6/\rho)\mathbf{a}_\phi\ \mu\text{T}}$$

Then, in the free space region, $\mathbf{B}(15 < \rho < 20) = \mu_0\mathbf{H} = \underline{(2.4/\rho)\mathbf{a}_\phi\ \mu\text{T}}$.

b) $0 < \phi < \pi$; Again, we are given $\mathbf{H} = 600/\pi\ \mathbf{a}_\phi$ A/m at $\rho = 10$ and at $\phi = \pi/2$. Now, since the interface between media lies in the $\mathbf{a}_\rho$ direction, and noting that magnetic field will be normal to this ($\mathbf{a}_\phi$ directed), we use the boundary condition of continuity of $\mathbf{B}$ normal to an interface, and write $\mathbf{B}(0 < \phi < \pi) = \mathbf{B}_1 = \mathbf{B}(\pi < \phi < 2\pi) = \mathbf{B}_2$, or $2.5\mu_0\mathbf{H}_1 = \mu_0\mathbf{H}_2$. Now, using Ampere's circuital law, we write

$$\oint \mathbf{H}\cdot d\mathbf{L} = \pi\rho H_1 + \pi\rho H_2 = 3.5\pi\rho H_1 = I$$

Using the given value for H_1 at $\rho = 10$ mm, $I = 3.5(600/\pi)(\pi \times 10^{-2}) = 21$ A. Therefore, $H_1 = 21/(3.5\pi\rho) = 6/(\pi\rho)$, or $\mathbf{H}(0 < \phi < \pi) = \underline{6/(\pi\rho)\,\mathbf{a}_\phi \text{ A/m}}$. Then $H_2 = 2.5H_1$, or $\mathbf{H}(\pi < \phi < 2\pi) = \underline{15/(\pi\rho)\,\mathbf{a}_\phi \text{ A/m}}$. Now $\mathbf{B}(0 < \phi < 2\pi) = 2.5\mu_0(6/(\pi\rho))\mathbf{a}_\phi = \underline{6/\rho\,\mathbf{a}_\phi\ \mu\text{T}}$. Now, in general, $\mathbf{M} = (\mu_R - 1)\mathbf{H}$, and so $\mathbf{M}(0 < \phi < \pi) = (2.5-1)6/(\pi\rho)\mathbf{a}_\phi = \underline{9/(\pi\rho)\,\mathbf{a}_\phi \text{ A/m}}$ and $\mathbf{M}(\pi < \phi < 2\pi) = \underline{0}$.

9.25. A conducting filament at $z = 0$ carries 12 A in the $\mathbf{a}_z$ direction. Let $\mu_R = 1$ for $\rho < 1$ cm, $\mu_R = 6$ for $1 < \rho < 2$ cm, and $\mu_R = 1$ for $\rho > 2$ cm. Find

a) $\mathbf{H}$ everywhere: This result will depend on the current and not the materials, and is:

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi = \underline{\frac{1.91}{\rho} \text{ A/m} \quad (0 < \rho < \infty)}$$

b) $\mathbf{B}$ everywhere: We use $\mathbf{B} = \mu_R\mu_0\mathbf{H}$ to find:

$\mathbf{B}(\rho < 1\,\text{cm}) = (1)\mu_0(1.91/\rho) = \underline{(2.4 \times 10^{-6}/\rho)\mathbf{a}_\phi \text{ T}}$
$\mathbf{B}(1 < \rho < 2\,\text{cm}) = (6)\mu_0(1.91/\rho) = \underline{(1.4 \times 10^{-5}/\rho)\mathbf{a}_\phi \text{ T}}$
$\mathbf{B}(\rho > 2\,\text{cm}) = (1)\mu_0(1.91/\rho) = \underline{(2.4 \times 10^{-6}/\rho)\mathbf{a}_\phi \text{ T}}$ where ρ is in meters.

9.26. Point $P(2, 3, 1)$ lies on the planar boundary boundary separating region 1 from region 2. The unit vector $\mathbf{a}_{N12} = 0.6\mathbf{a}_x + 0.48\mathbf{a}_y + 0.64\mathbf{a}_z$ is directed from region 1 to region 2. Let $\mu_{R1} = 2$, $\mu_{R2} = 8$, and $\mathbf{H}_1 = 100\mathbf{a}_x - 300\mathbf{a}_y + 200\mathbf{a}_z$ A/m. Find $\mathbf{H}_2$: First $\mathbf{B}_1 = 200\mu_0\mathbf{a}_x - 600\mu_0\mathbf{a}_y + 400\mu_0\mathbf{a}_z$. Then its normal component at the boundary will be $\mathbf{B}_{1N} = (\mathbf{B}_1 \cdot \mathbf{a}_{N12})\mathbf{a}_{N12} = (52.8\mathbf{a}_x + 42.24\mathbf{a}_y + 56.32\mathbf{a}_z)\mu_0 = \mathbf{B}_{2N}$. Then $\mathbf{H}_{2N} = \mathbf{B}_{2N}/(8\mu_0) = 6.60\mathbf{a}_x + 5.28\mathbf{a}_y + 7.04\mathbf{a}_z$, and $\mathbf{H}_{1N} = \mathbf{B}_{1N}/2\mu_0 = 26.40\mathbf{a}_x + 21.12\mathbf{a}_y + 28.16\mathbf{a}_z$. Now $\mathbf{H}_{1T} = \mathbf{H}_1 - \mathbf{H}_{1N} = (100\mathbf{a}_x - 300\mathbf{a}_y + 200\mathbf{a}_z) - (26.40\mathbf{a}_x + 21.12\mathbf{a}_y + 28.16\mathbf{a}_z) = 73.60\mathbf{a}_x - 321.12\mathbf{a}_y + 171.84\mathbf{a}_z = \mathbf{H}_{2T}$.

Finally, $\mathbf{H}_2 = \mathbf{H}_{2N} + \mathbf{H}_{2T} = \underline{80.2\mathbf{a}_x - 315.8\mathbf{a}_y + 178.9\mathbf{a}_z \text{ A/m}}$.

9.27. Let $\mu_{R1} = 2$ in region 1, defined by $2x + 3y - 4z > 1$, while $\mu_{R2} = 5$ in region 2 where $2x + 3y - 4z < 1$. In region 1, $\mathbf{H}_1 = 50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z$ A/m. Find:

a) $\mathbf{H}_{N1}$ (normal component of $\mathbf{H}_1$ at the boundary): We first need a unit vector normal to the surface, found through

$$\mathbf{a}_N = \frac{\nabla(2x + 3y - 4z)}{|\nabla(2x + 3y - 4z)|} = \frac{2\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z}{\sqrt{29}} = .37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z$$

Since this vector is found through the gradient, it will point in the direction of increasing values of $2x + 3y - 4z$, and so will be directed into region 1. Thus we write $\mathbf{a}_N = \mathbf{a}_{N21}$. The normal component of $\mathbf{H}_1$ will now be:

$$\begin{aligned}\mathbf{H}_{N1} &= (\mathbf{H}_1 \cdot \mathbf{a}_{N21})\mathbf{a}_{N21} \\ &= \left[(50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z) \cdot (.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z)\right](.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z) \\ &= \underline{-4.83\mathbf{a}_x - 7.24\mathbf{a}_y + 9.66\mathbf{a}_z \text{ A/m}}\end{aligned}$$

b) $\mathbf{H}_{T1}$ (tangential component of $\mathbf{H}_1$ at the boundary):

$$\begin{aligned}\mathbf{H}_{T1} &= \mathbf{H}_1 - \mathbf{H}_{N1} \\ &= (50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z) - (-4.83\mathbf{a}_x - 7.24\mathbf{a}_y + 9.66\mathbf{a}_z) \\ &= \underline{54.83\mathbf{a}_x - 22.76\mathbf{a}_y + 10.34\mathbf{a}_z \text{ A/m}}\end{aligned}$$

c) $\mathbf{H}_{T2}$ (tangential component of $\mathbf{H}_2$ at the boundary): Since tangential components of $\mathbf{H}$ are continuous across a boundary between two media of different permeabilities, we have

$$\mathbf{H}_{T2} = \mathbf{H}_{T1} = \underline{54.83\mathbf{a}_x - 22.76\mathbf{a}_y + 10.34\mathbf{a}_z \text{ A/m}}$$

d) $\mathbf{H}_{N2}$ (normal component of $\mathbf{H}_2$ at the boundary): Since normal components of $\mathbf{B}$ are continuous across a boundary between media of different permeabilities, we write $\mu_1\mathbf{H}_{N1} = \mu_2\mathbf{H}_{N2}$ or

$$\mathbf{H}_{N2} = \frac{\mu_{R1}}{\mu_R 2}\mathbf{H}_{N1} = \frac{2}{5}(-4.83\mathbf{a}_x - 7.24\mathbf{a}_y + 9.66\mathbf{a}_z) = \underline{-1.93\mathbf{a}_x - 2.90\mathbf{a}_y + 3.86\mathbf{a}_z \text{ A/m}}$$

e) θ_1, the angle between $\mathbf{H}_1$ and $\mathbf{a}_{N21}$: This will be

$$\cos\theta_1 = \frac{\mathbf{H}_1}{|\mathbf{H}_1|} \cdot \mathbf{a}_{N21} = \left[\frac{50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z}{(50^2 + 30^2 + 20^2)^{1/2}}\right] \cdot (.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z) = -0.21$$

Therefore $\theta_1 = \cos^{-1}(-.21) = \underline{102^\circ}$.

9.27f) θ_2, the angle between $\mathbf{H}_2$ and $\mathbf{a}_{N21}$: First,

$$\mathbf{H}_2 = \mathbf{H}_{T2} + \mathbf{H}_{N2} = (54.83\mathbf{a}_x - 22.76\mathbf{a}_y + 10.34\mathbf{a}_z) + (-1.93\mathbf{a}_x - 2.90\mathbf{a}_y + 3.86\mathbf{a}_z)$$
$$= 52.90\mathbf{a}_x - 25.66\mathbf{a}_y + 14.20\mathbf{a}_z \text{ A/m}$$

Now

$$\cos\theta_2 = \frac{\mathbf{H}_2}{|\mathbf{H}_2|} \cdot \mathbf{a}_{N21} = \left[\frac{52.90\mathbf{a}_x - 25.66\mathbf{a}_y + 14.20\mathbf{a}_z}{60.49}\right] \cdot (.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z) = -0.09$$

Therefore $\theta_2 = \cos^{-1}(-.09) = \underline{95^\circ}$.

9.28. For values of B below the knee on the magnetization curve for silicon steel, approximate the curve by a straight line with $\mu = 5$ mH/m. The core shown in Fig. 9.17 has areas of 1.6 cm^2 and lengths of 10 cm in each outer leg, and an area of 2.5 cm^2 and a length of 3 cm in the central leg. A coil of 1200 turns carrying 12 mA is placed around the central leg. Find B in the:

a) center leg: We use $mmf = \Phi R$, where, in the central leg,

$$R_c = \frac{L_{in}}{\mu A_{in}} = \frac{3 \times 10^{-2}}{(5 \times 10^{-3})(2.5 \times 10^{-4})} = 2.4 \times 10^4 \text{ H}$$

In each outer leg, the reluctance is

$$R_o = \frac{L_{out}}{\mu A_{out}} = \frac{10 \times 10^{-2}}{(5 \times 10^{-3})(1.6 \times 10^{-4})} = 1.25 \times 10^5 \text{ H}$$

The magnetic circuit is formed by the center leg in series with the parallel combination of the two outer legs. The total reluctance seen at the coil location is $R_T = R_c + (1/2)R_o = 8.65 \times 10^4$ H. We now have

$$\Phi = \frac{mmf}{R_T} = \frac{14.4}{8.65 \times 10^4} = 1.66 \times 10^{-4} \text{ Wb}$$

The flux density in the center leg is now

$$B = \frac{\Phi}{A} = \frac{1.66 \times 10^{-4}}{2.5 \times 10^{-4}} = \underline{0.666 \text{ T}}$$

b) center leg, if a 0.3-mm air gap is present in the center leg: The air gap reluctance adds to the total reluctance already calculated, where

$$R_{air} = \frac{0.3 \times 10^{-3}}{(4\pi \times 10^{-7})(2.5 \times 10^{-4})} = 9.55 \times 10^5 \text{ H}$$

Now the total reluctance is $R_{net} = R_T + R_{air} = 8.56 \times 10^4 + 9.55 \times 10^5 = 1.04 \times 10^6$. The flux in the center leg is now

$$\Phi = \frac{14.4}{1.04 \times 10^6} = 1.38 \times 10^{-5} \text{ Wb}$$

and

$$B = \frac{1.38 \times 10^{-5}}{2.5 \times 10^{-4}} = \underline{55.3 \text{ mT}}$$

9.29. In Problem 9.28, the linear approximation suggested in the statement of the problem leads to a flux density of 0.666 T in the center leg. Using this value of B and the magnetization curve for silicon steel, what current is required in the 1200-turn coil? With $B = 0.666$ T, we read $H_{in} \doteq 120$ A $\cdot$ t/m in Fig. 9.11. The flux in the center leg is $\Phi = 0.666(2.5 \times 10^{-4}) = 1.66 \times 10^{-4}$ Wb. This divides equally in the two outer legs, so that the flux density in each outer leg is

$$B_{out} = \left(\frac{1}{2}\right) \frac{1.66 \times 10^{-4}}{1.6 \times 10^{-4}} = 0.52 \text{ Wb/m}^2$$

Using Fig. 9.11 with this result, we find $H_{out} \doteq 90$ A $\cdot$ t/m We now use

$$\oint \mathbf{H} \cdot d\mathbf{L} = NI$$

to find

$$I = \frac{1}{N}\left(H_{in}L_{in} + H_{out}L_{out}\right) = \frac{(120)(3 \times 10^{-2}) + (90)(10 \times 10^{-2})}{1200} = \underline{10.5 \text{ mA}}$$

9.30. A toroidal core has a circular cross section of 4 cm^2 area. The mean radius of the toroid is 6 cm. The core is composed of two semi-circular segments, one of silicon steel and the other of a linear material with $\mu_R = 200$. There is a 4mm air gap at each of the two joints, and the core is wrapped by a 4000-turn coil carrying a dc current I_1.

a) Find I_1 if the flux density in the core is 1.2 T: I will use the reluctance method here. Reluctances of the steel and linear materials are respectively,

$$R_s = \frac{\pi(6 \times 10^{-2})}{(3.0 \times 10^{-3})(4 \times 10^{-4})} = 1.57 \times 10^5 \text{ H}^{-1}$$

$$R_l = \frac{\pi(6 \times 10^{-2})}{(200)(4\pi \times 10^{-7})(4 \times 10^{-4})} = 1.88 \times 10^6 \text{ H}^{-1}$$

where μ_s is found from Fig. 9.11, using $B = 1.2$, from which $H = 400$, and so $B/H = 3.0$ mH/m. The reluctance of each gap is now

$$R_g = \frac{0.4 \times 10^{-3}}{(4\pi \times 10^{-7})(4 \times 10^{-4})} = 7.96 \times 10^5 \text{ H}^{-1}$$

We now construct

$$NI_1 = \Phi R = 1.2(4 \times 10^{-4})\left[R_s + R_l + 2R_g\right] = 1.74 \times 10^3$$

Thus $I_1 = (1.74 \times 10^3)/4000 = \underline{435 \text{ mA}}$.

9.30b. Find the flux density in the core if $I_1 = 0.3$ A: We are not sure what to use for the permittivity of steel in this case, so we use the iterative approach. Since the current is down from the value obtained in part a, we can try $B = 1.0$ T and see what happens. From Fig. 9.11, we find $H = 200$ A/m. Then, in the linear material,

$$H_l = \frac{1.0}{200(4\pi \times 10^{-7})} = 3.98 \times 10^3 \text{ A/m}$$

and in each gap,

$$H_g = \frac{1.0}{4\pi \times 10^{-7}} = 7.96 \times 10^5 \text{ A/m}$$

Now Ampere's circuital law around the toroid becomes

$$NI_1 = \pi(.06)(200 + 3.98 \times 10^3) + 2(7.96 \times 10^5)(4 \times 10^{-4}) = 1.42 \times 10^3 \text{ A}-\text{t}$$

Then $I_1 = (1.42 \times 10^3)/4000 = .356$ A. This is still larger than the given value of .3A, so we can extrapolate down to find a better value for B:

$$B = 1.0 - (1.2 - 1.0)\left[\frac{.356 - .300}{.435 - .356}\right] = \underline{0.86 \text{ T}}$$

Using this value in the procedure above to evaluate Ampere's circuital law leads to a value of I_1 of 0.306 A. The result of 0.86 T for B is probably good enough for this problem, considering the limited resolution of Fig. 9.11.

9.31. A toroid is constructed of a magnetic material having a cross-sectional area of 2.5 cm^2 and an effective length of 8 cm. There is also a short air gap 0.25 mm length and an effective area of 2.8 cm^2. An mmf of 200 A $\cdot$ t is applied to the magnetic circuit. Calculate the total flux in the toroid if:

a) the magnetic material is assumed to have infinite permeability: In this case the core reluctance, $R_c = l/(\mu A)$, is zero, leaving only the gap reluctance. This is

$$R_g = \frac{d}{\mu_0 A_g} = \frac{0.25 \times 10^{-3}}{(4\pi \times 10^{-7})(2.5 \times 10^{-4})} = 7.1 \times 10^5 \text{ H}$$

Now

$$\Phi = \frac{mmf}{R_g} = \frac{200}{7.1 \times 10^5} = \underline{2.8 \times 10^{-4} \text{ Wb}}$$

b) the magnetic material is assumed to be linear with $\mu_R = 1000$: Now the core reluctance is no longer zero, but

$$R_c = \frac{8 \times 10^{-2}}{(1000)(4\pi \times 10^{-7})(2.5 \times 10^{-4})} = 2.6 \times 10^5 \text{ H}$$

The flux is then

$$\Phi = \frac{mmf}{R_c + R_g} = \frac{200}{9.7 \times 10^5} = \underline{2.1 \times 10^{-4} \text{ Wb}}$$

c) the magnetic material is silicon steel: In this case we use the magnetization curve, Fig. 9.11, and employ an iterative process to arrive at the final answer. We can begin with the value of Φ found in part a, assuming infinite permeability: $\Phi^{(1)} = 2.8 \times 10^{-4}$ Wb. The flux density in the core is then $B_c^{(1)} = (2.8 \times 10^{-4})/(2.5 \times 10^{-4}) = 1.1$ Wb/m^2. From Fig. 9.11, this corresponds to magnetic field strength $H_c^{(1)} \doteq 270$ A/m. We check this by applying Ampere's circuital law to the magnetic circuit:

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_c^{(1)} L_c + H_g^{(1)} d$$

where $H_c^{(1)} L_c = (270)(8 \times 10^{-2}) = 22$, and where $H_g^{(1)} d = \Phi^{(1)\cdot\cdot}{}_g = (2.8 \times 10^{-4})(7.1 \times 10^5) = 199$. But we require that

$$\oint \mathbf{H} \cdot d\mathbf{L} = 200 \text{ A} \cdot \text{t}$$

whereas the actual result in this first calculation is $199 + 22 = 221$, which is too high. So, for a second trial, we reduce B to $B_c^{(2)} = 1$ Wb/m^2. This yields $H_c^{(2)} = 200$ A/m from Fig. 9.11, and thus $\Phi^{(2)} = 2.5 \times 10^{-4}$ Wb. Now

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_c^{(2)} L_c + \Phi^{(2)} R_g = 200(8 \times 10^{-2}) + (2.5 \times 10^{-4})(7.1 \times 10^5) = 194$$

This is less than 200, meaning that the actual flux is $\underline{\text{slightly higher than } 2.5 \times 10^{-4} \text{ Wb}}$. I will leave the answer at that, considering the lack of fine resolution in Fig. 9.11.

9.32. Determine the total energy stored in a spherical region 1cm in radius, centered at the origin in free space, in the uniform field:

a) $\mathbf{H}_1 = -600\mathbf{a}_y$ A/m: First we find the energy density:

$$w_{m1} = \frac{1}{2}\mathbf{B}_1 \cdot \mathbf{H}_1 = \frac{1}{2}\mu_0 H_1^2 = \frac{1}{2}(4\pi \times 10^{-7})(600)^2 = 0.226 \text{ J/m}^3$$

The energy within the sphere is then

$$W_{m1} = w_{m1}\left(\frac{4}{3}\pi a^3\right) = 0.226\left(\frac{4}{3}\pi \times 10^{-6}\right) = \underline{0.947\ \mu\text{J}}$$

b) $\mathbf{H}_2 = 600\mathbf{a}_x + 1200\mathbf{a}_y$ A/m: In this case the energy density is

$$w_{m2} = \frac{1}{2}\mu_0\left[(600)^2 + (1200)^2\right] = \frac{5}{2}\mu_0(600)^2$$

or five times the energy density that was found in part a. Therefore, the stored energy in this field is five times the amount in part a, or $W_{m2} = \underline{4.74\ \mu\text{J}}$.

c) $\mathbf{H}_3 = -600\mathbf{a}_x + 1200\mathbf{a}_y$. This field differs from $\mathbf{H}_2$ only by the negative x component, which is a non-issue since the component is squared when finding the energy density. Therefore, the stored energy will be the same as that in part b, or $W_{m3} = \underline{4.74\ \mu\text{J}}$.

d) $\mathbf{H}_4 = \mathbf{H}_2 + \mathbf{H}_3$, or $2400\mathbf{a}_y$ A/m: The energy density is now $w_{m4} = (1/2)\mu_0(2400)^2 = (1/2)\mu_0(16)(600)^2$ J/m^3, which is sixteen times the energy density in part a. The stored energy is therefore sixteen times that result, or $W_{m4} = 16(0.947) = \underline{15.2\ \mu\text{J}}$.

e) $1000\mathbf{a}_x$ A/m$+0.001\mathbf{a}_x$ T: The energy density is $w_{m5} = (1/2)\mu_0[1000 + .001/\mu_0]^2 = 2.03$ J/m^3. Then $W_{m5} = 2.03[(4/3)\pi \times 10^{-6}] = \underline{8.49\ \mu\text{J}}$.

9.33. A toroidal core has a square cross section, 2.5 cm $< \rho <$ 3.5 cm, -0.5 cm $< z <$ 0.5 cm. The upper half of the toroid, $0 < z < 0.5$ cm, is constructed of a linear material for which $\mu_R = 10$, while the lower half, -0.5 cm $< z < 0$, has $\mu_R = 20$. An mmf of 150 A $\cdot$ t establishes a flux in the $\mathbf{a}_\phi$ direction. For $z > 0$, find:

a) $H_\phi(\rho)$: Ampere's circuital law gives:

$$2\pi\rho H_\phi = NI = 150 \;\Rightarrow\; H_\phi = \frac{150}{2\pi\rho} = \underline{23.9/\rho \text{ A/m}}$$

b) $B_\phi(\rho)$: We use $B_\phi = \mu_R\mu_0 H_\phi = (10)(4\pi \times 10^{-7})(23.9/\rho) = \underline{3.0 \times 10^{-4}/\rho \text{ Wb/m}^2}$.

c) $\Phi_{z>0}$: This will be

$$\Phi_{z>0} = \int\int \mathbf{B}\cdot d\mathbf{S} = \int_0^{.005}\int_{.025}^{.035} \frac{3.0\times 10^{-4}}{\rho}\, d\rho dz = (.005)(3.0\times 10^{-4})\ln\left(\frac{.035}{.025}\right)$$
$$= \underline{5.0 \times 10^{-7} \text{ Wb}}$$

d) Repeat for $z < 0$: First, the magnetic field strength will be the same as in part a, since the calculation is material-independent. Thus $H_\phi = \underline{23.9/\rho \text{ A/m}}$. Next, B_ϕ is modified only by the new permeability, which is twice the value used in part a: Thus $B_\phi = \underline{6.0 \times 10^{-4}/\rho \text{ Wb/m}^2}$. Finally, since B_ϕ is twice that of part a, the flux will be increased by the same factor, since the area of integration for $z < 0$ is the same. Thus $\Phi_{z<0} = \underline{1.0 \times 10^{-6} \text{ Wb}}$.

e) Find Φ_{total}: This will be the sum of the values found for $z < 0$ and $z > 0$, or $\Phi_{\text{total}} = \underline{1.5 \times 10^{-6} \text{ Wb}}$.

9.34. Three planar current sheets are located in free space as follows: $-100\mathbf{a}_x$ A/m^2 at $z = -1$, $200\mathbf{a}_x$ A/m^2 at $z = 0$, $-100\mathbf{a}_x$ A/m^2 at $z = 1$. Let $w_H = (1/2)\mathbf{B}\cdot\mathbf{H}$ J/m^3, and find w_H for all z: Using the fact that the field on either side of a current sheet is given in magnitude by $H = K/2$, we find, in A/m:

$$\mathbf{H}(z > 1) = (1/2)(-200 + 100 + 100)\mathbf{a}_y = 0$$

$$\mathbf{H}(0 < z < 1) = (1/2)(-200 - 100 + 100)\mathbf{a}_y = -100\mathbf{a}_y$$

$$\mathbf{H}(-1 < z < 0) = (1/2)(200 - 100 + 100)\mathbf{a}_y = 100\mathbf{a}_y$$

and

$$\mathbf{H}(z < -1) = (1/2)(200 - 100 - 100)\mathbf{a}_y = 0$$

The energy densities are then

$$w_H(z > 1) = w_H(z < -1) = \underline{0}$$
$$w_H(0 < z < 1) = w_H(-1 < z < 0) = (1/2)\mu_0(100)^2 = \underline{6.28 \text{ mJ/m}^2}$$

9.35. The cones $\theta = 21^\circ$ and $\theta = 159^\circ$ are conducting surfaces and carry total currents of 40 A, as shown in Fig. 9.18. The currents return on a spherical conducting surface of 0.25 m radius.

a) Find **H** in the region $0 < r < 0.25$, $21^\circ < \theta < 159^\circ$, $0 < \phi < 2\pi$: We can apply Ampere's circuital law and take advantage of symmetry. We expect to see **H** in the $\mathbf{a}_\phi$ direction and it would be constant at a given distance from the z axis. We thus perform the line integral of **H** over a circle, centered on the z axis, and parallel to the xy plane:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \mathbf{a}_\phi \cdot r \sin\theta \mathbf{a}_\phi \, d\phi = I_{encl.} = 40 \text{ A}$$

Assuming that H_ϕ is constant over the integration path, we take it outside the integral and solve:

$$H_\phi = \frac{40}{2\pi r \sin\theta} \quad \Rightarrow \quad \mathbf{H} = \underline{\frac{20}{\pi r \sin\theta} \mathbf{a}_\phi \text{ A/m}}$$

b) How much energy is stored in this region? This will be

$$W_H = \int_v \frac{1}{2}\mu_0 H_\phi^2 = \int_0^{2\pi}\int_{21^\circ}^{159^\circ}\int_0^{.25} \frac{200\mu_0}{\pi^2 r^2 \sin^2\theta} r^2 \sin\theta \, dr\, d\theta\, d\phi = \frac{100\mu_0}{\pi}\int_{21^\circ}^{159^\circ} \frac{d\theta}{\sin\theta}$$
$$= \frac{100\mu_0}{\pi} \ln\left[\frac{\tan(159/2)}{\tan(21/2)}\right] = \underline{1.35 \times 10^{-4} \text{ J}}$$

9.36. A filament carrying current I in the $\mathbf{a}_z$ direction lies on the z axis, and cylindrical current sheets of $5\mathbf{a}_z$ A/m and $-2\mathbf{a}_z$ A/m are located at $\rho = 3$ and $\rho = 10$, respectively.

a) Find I if $\mathbf{H} = 0$ for $\rho > 10$. Ampere's circuital law says, for $\rho > 10$:

$$2\pi\rho H = 2\pi(3)(5) - 2\pi(10)(2) + I = 0$$

from which $I = 2\pi(10)(3) - 2\pi(3)(5) = \underline{10\pi \text{ A}}$.

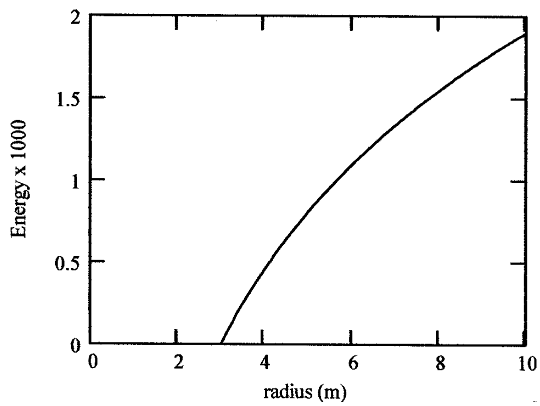
b) Using this value of I, calculate **H** for all ρ, $3 < \rho < 10$: Again, using Ampere's circuital law, we find

$$\mathbf{H}(3 < \rho < 10) = \frac{1}{2\pi\rho}\left[10\pi + 2\pi(3)(5)\right]\mathbf{a}_\phi = \underline{\frac{20}{\rho}\mathbf{a}_\phi \text{ A/m}}$$

c) Calculate and plot W_H versus ρ_0, where W_H is the total energy stored within the volume $0 < z < 1$, $0 < \phi < 2\pi$, $3 < \rho < \rho_0$: First the energy density will be $w_H = (1/2)\mu_0 H^2 = 200\mu_0/\rho^2$ J/m^3. Then the energy is

$$W_H = \int_0^1\int_0^{2\pi}\int_3^{\rho_0} \frac{200\mu_0}{\rho^2}\rho\, d\rho\, d\phi\, dz = 400\pi\mu_0 \ln\left(\frac{\rho_0}{3}\right) = \underline{(1.58 \times 10^{-3}) \ln\left(\frac{\rho_0}{3}\right) \text{ J}}$$

9.36c. (continued) A plot of the energy as a function of ρ_0 is shown below.



9.37. Find the inductance of the cone-sphere configuration described in Problem 9.35 and Fig. 9.18. The inductance is that offered at the origin between the vertices of the cone: From Problem 9.35, the magnetic flux density is $B_\phi = 20\mu_0/(\pi r \sin\theta)$. We integrate this over the crossectional area defined by $0 < r < 0.25$ and $21^\circ < \theta < 159^\circ$, to find the total flux:

$$\Phi = \int_{21^\circ}^{159^\circ} \int_0^{0.25} \frac{20\mu_0}{\pi r \sin\theta}\, r\, dr\, d\theta = \frac{5\mu_0}{\pi} \ln\left[\frac{\tan(159/2)}{\tan(21/2)}\right] = \frac{5\mu_0}{\pi}(3.37) = 6.74 \times 10^{-6} \text{ Wb}$$

Now $L = \Phi/I = 6.74 \times 10^{-6}/40 = \underline{0.17\ \mu\text{H}}$.
Second method: Use the energy computation of Problem 9.35, and write

$$L = \frac{2W_H}{I^2} = \frac{2(1.35 \times 10^{-4})}{(40)^2} = \underline{0.17\ \mu\text{H}}$$

9.38. A toroidal core has a rectangular cross section defined by the surfaces $\rho = 2$ cm, $\rho = 3$ cm, $z = 4$ cm, and $z = 4.5$ cm. The core material has a relative permeability of 80. If the core is wound with a coil containing 8000 turns of wire, find its inductance: First we apply Ampere's circuital law to a circular loop of radius ρ in the interior of the toroid, and in the $\mathbf{a}_\phi$ direction.

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = NI \quad \Rightarrow \quad H_\phi = \frac{NI}{2\pi\rho}$$

The flux in the toroid is then the integral over the cross section of $\mathbf{B}$:

$$\Phi = \int\int \mathbf{B} \cdot d\mathbf{L} = \int_{.04}^{.045} \int_{.02}^{.03} \frac{\mu_R \mu_0 NI}{2\pi\rho}\, d\rho\, dz = (.005)\frac{\mu_R \mu_0 NI}{2\pi} \ln\left(\frac{.03}{.02}\right)$$

The flux linkage is then given by $N\Phi$, and the inductance is

$$L = \frac{N\Phi}{I} = \frac{(.005)(80)(4\pi \times 10^{-7})(8000)^2}{2\pi} \ln(1.5) = \underline{2.08 \text{ H}}$$

9.39. Conducting planes in air at $z = 0$ and $z = d$ carry surface currents of $\pm K_0 \mathbf{a}_x$ A/m.

a) Find the energy stored in the magnetic field per unit length ($0 < x < 1$) in a width w ($0 < y < w$): First, assuming current flows in the $+\mathbf{a}_x$ direction in the sheet at $z = d$, and in $-\mathbf{a}_x$ in the sheet at $z = 0$, we find that both currents together yield $\mathbf{H} = K_0 \mathbf{a}_y$ for $0 < z < d$ and zero elsewhere. The stored energy within the specified volume will be:

$$W_H = \int_v \frac{1}{2}\mu_0 H^2 dv = \int_0^d \int_0^w \int_0^1 \frac{1}{2}\mu_0 K_0^2 \, dx \, dy \, dz = \underline{\frac{1}{2} w d \mu_0 K_0^2 \text{ J/m}}$$

b) Calculate the inductance per unit length of this transmission line from $W_H = (1/2)LI^2$, where I is the total current in a width w in either conductor: We have $I = wK_0$, and so

$$L = \frac{2}{I^2}\frac{wd}{2}\mu_0 K_0^2 = \frac{2}{w^2 K_0^2}\frac{dw}{2}\mu_0 K_0^2 = \underline{\frac{\mu_0 d}{w} \text{ H/m}}$$

c) Calculate the total flux passing through the rectangle $0 < x < 1$, $0 < z < d$, in the plane $y = 0$, and from this result again find the inductance per unit length:

$$\Phi = \int_0^d \int_0^1 \mu_0 H \mathbf{a}_y \cdot \mathbf{a}_y \, dx \, dz = \int_0^d \int_0^1 \mu_0 K_0 dx \, dy = \mu_0 d K_0$$

Then

$$L = \frac{\Phi}{I} = \frac{\mu_0 d K_0}{w K_0} = \underline{\frac{\mu_0 d}{w} \text{ H/m}}$$

9.40. A coaxial cable has conductor dimensions of 1 and 5 mm. The region between conductors is air for $0 < \phi < \pi/2$ and $\pi < \phi < 3\pi/2$, and a non-conducting material having $\mu_R = 8$ for $\pi/2 < \phi < \pi$ and $3\pi/2 < \phi < 2\pi$. Find the inductance per meter length: The interfaces between media all occur along radial lines, normal to the direction of **B** and **H** in the coax line. **B** is therefore continuous (and constant at constant radius) around a circular loop centered on the z axis. Ampere's circuital law can thus be written in this form:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \frac{B}{\mu_0}\left(\frac{\pi}{2}\rho\right) + \frac{B}{\mu_R \mu_0}\left(\frac{\pi}{2}\rho\right) + \frac{B}{\mu_0}\left(\frac{\pi}{2}\rho\right) + \frac{B}{\mu_R \mu_0}\left(\frac{\pi}{2}\rho\right) = \frac{\pi \rho B}{\mu_R \mu_0}(\mu_R + 1) = I$$

and so

$$\mathbf{B} = \frac{\mu_R \mu_0 I}{\pi \rho (1 + \mu_R)} \mathbf{a}_\phi$$

The flux in the line per meter length in z is now

$$\Phi = \int_0^1 \int_{.001}^{.005} \frac{\mu_R \mu_0 I}{\pi \rho (1 + \mu_R)} \, d\rho \, dz = \frac{\mu_R \mu_0 I}{\pi (1 + \mu_R)} \ln(5)$$

And the inductance per unit length is:

$$L = \frac{\Phi}{I} = \frac{\mu_R \mu_0}{\pi (1 + \mu_R)} \ln(5) = \frac{8(4\pi \times 10^{-7})}{\pi(9)} \ln(5) = \underline{572 \text{ nH/m}}$$

9.41. A rectangular coil is composed of 150 turns of a filamentary conductor. Find the mutual inductance in free space between this coil and an infinite straight filament on the z axis if the four corners of the coil are located at

a) (0,1,0), (0,3,0), (0,3,1), and (0,1,1): In this case the coil lies in the yz plane. If we assume that the filament current is in the $+\mathbf{a}_z$ direction, then the $\mathbf{B}$ field from the filament penetrates the coil in the $-\mathbf{a}_x$ direction (normal to the loop plane). The flux through the loop will thus be

$$\Phi = \int_0^1 \int_1^3 \frac{-\mu_0 I}{2\pi y} \mathbf{a}_x \cdot (-\mathbf{a}_x)\, dy\, dz = \frac{\mu_0 I}{2\pi} \ln 3$$

The mutual inductance is then

$$M = \frac{N\Phi}{I} = \frac{150\mu_0}{2\pi} \ln 3 = \underline{33\ \mu\text{H}}$$

b) (1,1,0), (1,3,0), (1,3,1), and (1,1,1): Now the coil lies in the $x = 1$ plane, and the field from the filament penetrates in a direction that is not normal to the plane of the coil. We write the $\mathbf{B}$ field from the filament at the coil location as

$$\mathbf{B} = \frac{\mu_0 I \mathbf{a}_\phi}{2\pi\sqrt{y^2+1}}$$

The flux through the coil is now

$$\Phi = \int_0^1 \int_1^3 \frac{\mu_0 I \mathbf{a}_\phi}{2\pi\sqrt{y^2+1}} \cdot (-\mathbf{a}_x)\, dy\, dz = \int_0^1 \int_1^3 \frac{\mu_0 I \sin\phi}{2\pi\sqrt{y^2+1}}\, dy\, dz$$

$$= \int_0^1 \int_1^3 \frac{\mu_0 I y}{2\pi(y^2+1)}\, dy\, dz = \frac{\mu_0 I}{2\pi} \ln(y^2+1)\Big|_1^3 = (1.6 \times 10^{-7}) I$$

The mutual inductance is then

$$M = \frac{N\Phi}{I} = (150)(1.6 \times 10^{-7}) = \underline{24\ \mu\text{H}}$$

9.42. Find the mutual inductance of this conductor system in free space:

a) the solenoid of Fig. 8.11b and a square filamentary loop of side length b coaxially centered inside the solenoid, if $a > b/\sqrt{2}$; With the given side length, the loop lies entirely inside the solenoid, and so is linked over its entire cross section by the solenoid field. The latter is given by $\mathbf{B} = \mu_0 NI/d\ \mathbf{a}_z$ T. The flux through the loop area is now $\Phi = Bb^2$, and the mutual inductance is $M = \Phi/I = \underline{\mu_0 Nb^2/d\ \text{H}}$.

b) a cylindrical conducting shell of a radius a, axis on the z axis, and a filament at $x = 0$, $y = d$, *and where* $d > a$ (omitted from problem statement); The $\mathbf{B}$ field from the cylinder is $\mathbf{B} = (\mu_0 I)/(2\pi\rho)\, \mathbf{a}_\phi$ for $\rho > a$, and so the flux per unit length between cylinder and wire is

$$\Phi = \int_0^1 \int_a^d \frac{\mu_0 I}{2\pi\rho}\, d\rho\, dz = \frac{\mu_0 I}{2\pi} \ln\left(\frac{d}{a}\right) \text{ Wb}$$

Finally the mutual inductance is $M = \Phi/I = \underline{\mu_0/2\pi\ \ln(d/a)\ \text{H}}$.

9.43. a) Use energy relationships to show that the internal inductance of a nonmagnetic cylindrical wire of radius a carrying a uniformly-distributed current I is $\mu_0/(8\pi)$ H/m. We first find the magnetic field inside the conductor, then calculate the energy stored there. From Ampere's circuital law:

$$2\pi\rho H_\phi = \frac{\pi\rho^2}{\pi a^2} I \quad \Rightarrow \quad H_\phi = \frac{I\rho}{2\pi a^2} \text{ A/m}$$

Now

$$W_H = \int_v \frac{1}{2}\mu_0 H_\phi^2 \, dv = \int_0^1 \int_0^{2\pi} \int_0^a \frac{\mu_0 I^2 \rho^2}{8\pi^2 a^4} \, \rho \, d\rho \, d\phi \, dz = \frac{\mu_0 I^2}{16\pi} \text{ J/m}$$

Now, with $W_H = (1/2)LI^2$, we find $L_{int} = \mu_0/(8\pi)$ as expected.

b) Find the internal inductance if the portion of the conductor for which $\rho < c < a$ is removed: The hollowed-out conductor still carries current I, so Ampere's circuital law now reads:

$$2\pi\rho H_\phi = \frac{\pi(\rho^2 - c^2)}{\pi(a^2 - c^2)} \quad \Rightarrow \quad H_\phi = \frac{I}{2\pi\rho}\left[\frac{\rho^2 - c^2}{a^2 - c^2}\right] \text{ A/m}$$

and the energy is now

$$W_H = \int_0^1 \int_0^{2\pi} \int_c^a \frac{\mu_0 I^2 (\rho^2 - c^2)^2}{8\pi^2 \rho^2 (a^2 - c^2)^2} \, \rho \, d\rho \, d\phi \, dz = \frac{\mu_0 I^2}{4\pi(a^2 - c^2)^2} \int_c^a \left[\rho^3 - 2c^2\rho + \frac{C^4}{\rho}\right] d\rho$$

$$= \frac{\mu_0 I^2}{4\pi(a^2 - c^2)^2} \left[\frac{1}{4}(a^4 - c^4) - c^2(a^2 - c^2) + c^4 \ln\left(\frac{a}{c}\right)\right] \text{ J/m}$$

The internal inductance is then

$$L_{int} = \frac{2W_H}{I^2} = \underline{\frac{\mu_0}{8\pi}\left[\frac{a^4 - 4a^2c^2 + 3c^4 + 4c^4 \ln(a/c)}{(a^2 - c^2)^2}\right] \text{ H/m}}$$