18-2-20

# SIGNALS AND SYSTEMS.

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Google classroom: Signals and systems spring 2020.

Code: evjx3d6

Signals and Systems By Oppenheim.

## Signal:

Signal is anything that is a function of one or more independent variables and it typically carries some information.

One independent variable ⇒ One dimensional.
Two " " ⇒ Two dimensional.

Signal is not necessarily time dependent.
eg Temperature is a signal that depends on height (distance)
→ Image is a two dimensional signal depending on the position (x and y cordinates) (brightness at different points).

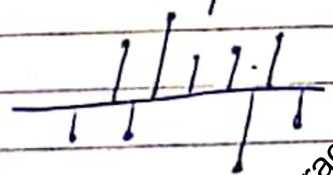
Here we will mostly deal with one dimensional signals and assume that they are time dependent.

## Two types of signals:

1. Continous time signals.
2. Discrete time signals.

Continous time signal. is a signal That is defined at any point of time. (which varys continously w.r.t time).
Represented as $x(t)$.

Discrete time signal is a signal That is defined only for a discrete value of time.
Represented as $x[n]$

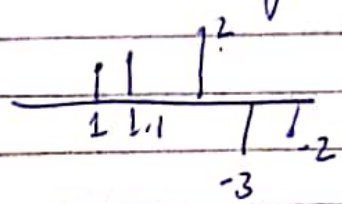


n is also representing time
But a particular value of time.

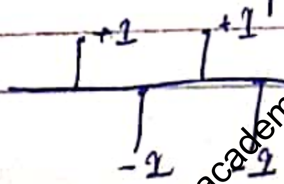
Similarly we have another category ie analog and digital signal.

Analog signal is a signal that can take any value. (continous domain).

eg this is a discrete time analogue signal.

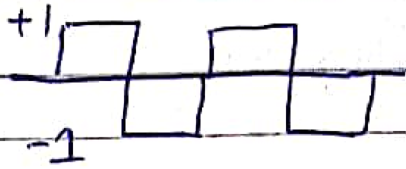



Digital signal is a signal that can only take finite number of values.




discrete time digital

continous time digital




⇒ Similarly we have periodic and non periodic signals.

A ~~pe~~ signal is periodic if;

$$x(t) = x(t+T)$$

for any value of $t$.

If this condition is not fulfilled for any value of $t$, so the function is non periodic.

$T \rightarrow$ fundamental periodic $\rightarrow$ the smallest value of $T$ for which function is periodic.

For a periodic discrete time signal,

$$x[n] = x[n+N]$$

⇒ We also have determinstic and random signals.

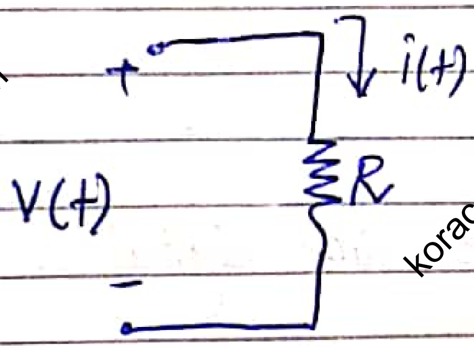
Deterministic signal is a signal whose value can be predicted.
It always has a mathematical equation.

Consider $x(t) = \sin \omega t$ continous
$x[n] = \sin \omega n$ discrete

$\sin \omega t$ is periodic whereas $\sin \omega n$ may or may not be periodic.
↳ periodic for a particular value of $\omega$.

## Lecture 2 25/02/20

### Power Signal / Energy Signal.



$$P(t) = V(t) \times i(t)$$

or

$$P(t) = \frac{V^2(t)}{R}$$

or

Instantaneous power ← $P(t) = I^2(t) R$

Energy dissipated during time interval;
$t_1 \leq t \leq t_2$

$$E = \int_{t_1}^{t_2} p(t)\, dt$$

Average power;

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} P(t)\, dt$$

CT → $x(t)$ D.T → $x[n]$.

Generally complex

$$E = \int_{t_1}^{t_2} |x(t)|^2 dt$$

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt$$

$$E = \sum_{n_1}^{n_2} |x[n]|^2 \qquad n_1 \le n \le n_2$$

$$P = \frac{1}{n_2 - n_1 + 1} \sum_{n_1}^{n_2} |x[n]|^2$$

The above formulas are for finite interval of time;

If For infinite interval ie,

$$-\infty \le t \le \infty$$

$$E_\infty = \lim_{T \to \infty} \int_{-T}^{+T} |x(t)|^2 dt$$

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 dt$$

Similarly for discrete time signals;

if $-\infty \le n \le \infty$

$$E_\infty = \lim_{N \to \infty} \sum_{n=-N}^{+N} |x[n]|^n$$

$$P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

A signal is an energy signal if;

$$E_\infty < \infty$$

Similarly a signal is power signal if;

$$P_\infty < \infty$$

## Signal Transformation

① Time Shift

$$x(t) = x(t - t_0)$$
$$x[n] \to x[n - n_0]$$

If $t_0$ is +ve → delay

If $t_0$ is -ve → advance

② Time reversal

$$x(t) \to x(-t)$$
$$x[n] \to x[-n]$$

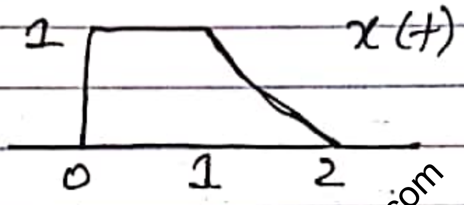
The signal will flip around the vertical axis.

③ Time Scaling

$$x(t) \rightarrow x(at)$$

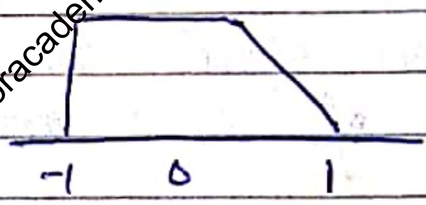
If $|a| > 1$ → compression.
If $|a| < 1$ → expand

Example.

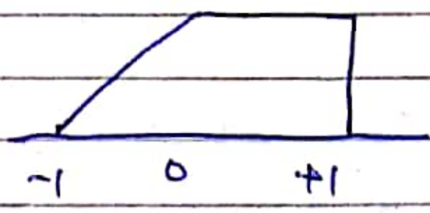



$x\left(-\frac{3t}{2}+1\right) = ?$

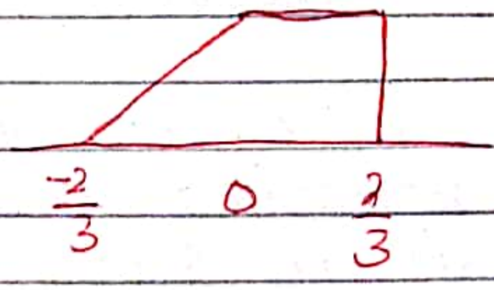
$x(t+1) \rightarrow$




$x(-t+1) \rightarrow$




$x\left(-\frac{3}{2}t+1\right)$



## Even signal / odd signal:

A signal is even if after applying time reversal, the signal shape is same.

$$x(t) = x(-t)$$
$$x[n] = x[-n]$$

A signal is odd if;

$$x(t) = -x(-t)$$
$$x[n] = -x[-n]$$

Any given signal can be represented as a sum of even and odd signal.

$$x(t) = Ev\{x(t)\} + Odd\{x(t)\}$$

$$Ev\{x(t)\} = \frac{1}{2}\{x(t) + x(-t)\}$$

$$Odd\{x(t)\} = \frac{1}{2}\{x(t) - x(-t)\}$$

Similarly a discrete time signal can also be represented as a sum of even and odd signals.

$$x[n] = Even\{x[n]\} + Odd\{x[n]\}$$

# Basic Signals.

## ① Exponential and Sinosidal signals:

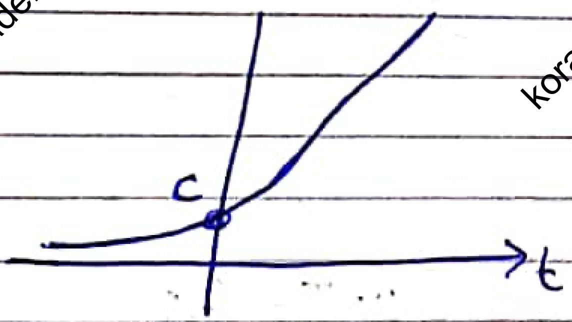
First we see them in continous time domain.

$$x(t) = Ce^{at}$$

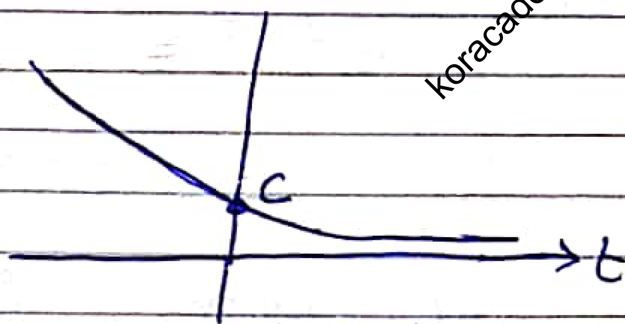
C and a are generally complex.

Case 1 If c and a are real.
↳ Real exponential signal.

$a > 0$ $a < 0$




exponential growth




exponential decay

c is the value of function at $t = 0$

If $a = 0 \rightarrow x(t) = c$
Constant function is a special case of real exponential function.

Case 2 $c = 1$ and a is purely imaginary.
Say $a = j\omega_0$

$$x(t) = e^{j\omega_0 t}$$

## Complex periodic exponential signal

It is very much similar to a sinosidal signal.

$$x(t) = A\cos(w_0 t + \phi)$$

$$= Re\{A e^{j(w_0 t + \phi)}\}$$

Why periodic?

As for periodic $x(t) = x(t+T)$

$$e^{jw_0 t} = e^{jw_0(t+T)}$$

$$\cancel{e^{jw_0 t}} = \cancel{e^{jw_0 t}} \cdot e^{jw_0 T}$$

$$1 = e^{jw_0 T}$$

The condition is fulfilled if $w_0 T = 2\pi$

As $e^{j2\pi} = 1$

fundamental period $\leftarrow$ $T_0 = \dfrac{2\pi}{|w_0|}$

Suppose $x(t)$ is periodic with fundamental ~~freq~~ period $T_0$.

$$x(t) = e^{jwt}$$

$$x(t) = x(t + T_0)$$

$$e^{jwt} = e^{jwt} e^{jwT_0}$$

$$1 = e^{j\omega T_0}$$

Condition satisfies if $\omega T_0 = 2\pi K$

$$\omega = \omega_0 K \Leftarrow \omega = \frac{2\pi}{T_0} K$$

where $K = 0, \pm 1, \pm 2, \pm 3, \ldots$

$e^{jK\omega_0 t}$ → Harmonically related exponential signals.

$K=0$ Constant signal → fundamental $f = 0$
period → undefined.

$K=2$ $f$ will be $2\omega_0$.
fundamental period will be $T_0/2$

$K=1$ $\omega_0$ → frequency fund period → $T_0$

→ $T_0$ is its period but not fundamental.

∴ Every signal for every value of $K$ will be a new signal but have one period $T_0$ in common.

Case 3 $c$ and $a$ are both complex.

$$c = |c| e^{j\theta}$$

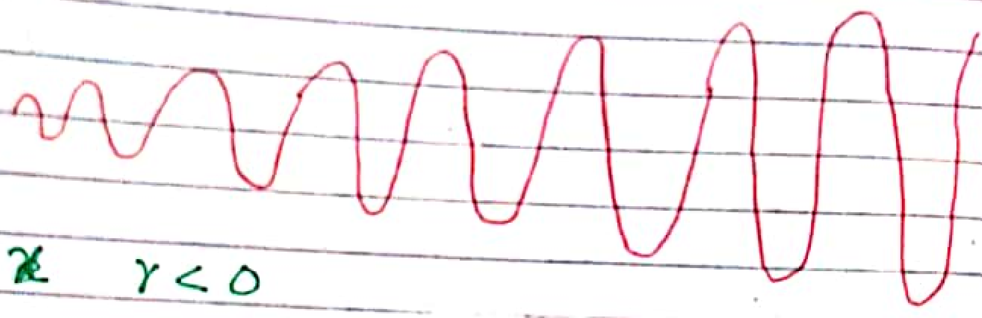
$$a = r + j\omega_0$$

$$x(t) = |c| e^{j\theta} \, e^{(r+j\omega_0)t}$$

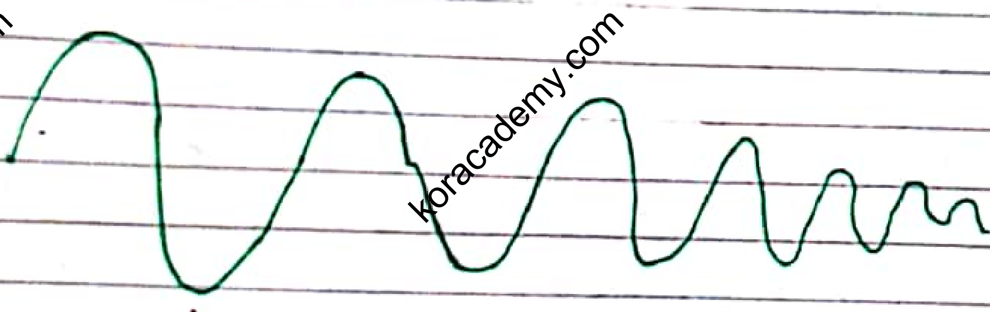
$$x(t) = |c| e^{rt} \, e^{j(\omega_0 t+\theta)t}$$

→ This is a real exponential multiplied with a periodic signal.

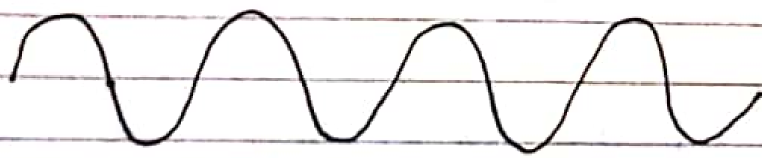
$r>0$



$r<0$



$r=0$



Next we consider discrete time signals.

$$x[n] = C\alpha^n$$

or $\quad x[n] = Ce^{j\beta n} \quad$ where $\alpha = e^{j\beta}$

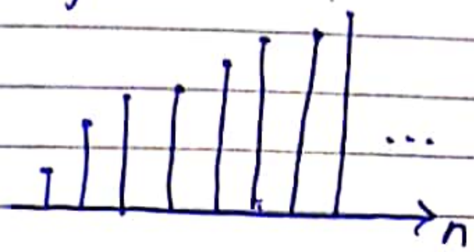
$C$ and $\alpha$ are generally complex.

Case 1  c and $\alpha$ are both real.

$$x[n] = c\alpha^n$$

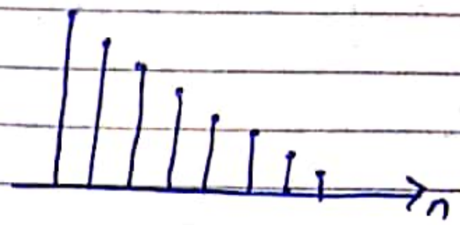
Is real exponential signal.

If $|\alpha| > 1$



exp growth

If $|\alpha| < 1$

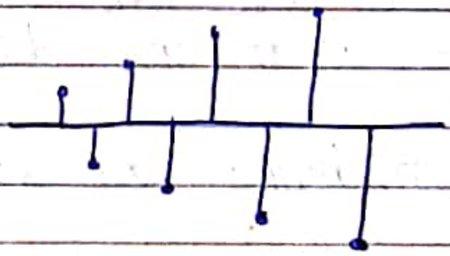


exp decay.

Say $\alpha = +2$ same

$\alpha = -2$

$\alpha = (-2)^1, (-2)^2, (-2)^3 \leftarrow$ ....



If $|\alpha| = 1 \rightarrow$ constant function $= c$.

Case 2  $c = 1$  $\beta$ = imaginary $= jw_0$

$$x[n] = e^{jw_0 n}$$

Complex periodic exponential signal

$\Downarrow$

not always periodic (discussed later)

$\rightarrow$ Closely related to sinosidal signals

$$x[n] = A\cos(w_0 n + \phi)$$

<u>Case 3</u> c and $\alpha$ are complex.

$$c = |c| e^{j\theta} \qquad \alpha = |\alpha| e^{jw_0}$$

$$x[n] = |c| e^{j\theta} |\alpha|^n e^{jw_0 n}$$

$$x[n] = |c| |\alpha|^n e^{j(w_0 n + \theta)}$$

If $\alpha > 1$ → growing sinosidal signal.

If $\alpha < 1$ → decaying sinosidal signal.

If $\alpha = 1$ → constant sinosidal signal

We have similarities in C.T and D.T but what are differences?

Considering case 2;

$$x(t) = e^{jw_0 t}$$

and $x[n] = e^{jw_0 n}$ ↳ its periodicity is within a certain finite range.

↳ For any values of $\omega_0$ it is periodic.

↳ Say increase frequency from $w_0$ to $w_0 + 2\pi$.

$$e^{j(w_0 + 2\pi)n} = e^{jw_0 n} \cdot e^{j2\pi n} = e^{jw_0 n}$$

So we did not get a new signal with a higher frequency; but we got the same signal back.

So here $\quad 0 \le \omega_0 \le 2\pi$

or $\quad -\pi \le \omega_0 \le \pi$

Frequency is maximum at $\omega_0 = \pi$

Signals whose frequency is near to $\pi$ is a high frequency signal and signal whose frequency is near to 0 to $2\pi$ is a low frequency signal.

Frequency increases from 0 to $\pi$, then decreases from $\pi$ to $2\pi$ and repeats in such a way.

Condition for periodicity;

$$x[n] = x[n+N]$$

$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)}$$

$$\cancel{e^{j\omega_0 n}} = \cancel{e^{j\omega_0 n}} \cdot e^{j\omega_0 N}$$

$$1 = e^{j\omega_0 N}$$

This is only true if;

$$\omega_0 N = 2\pi m$$

$$m = 0, \pm 1, \pm 2, \pm 3, \dots$$

$\frac{\omega_0}{2\pi} = \frac{m}{N}$ → condition in discrete domain.

$\frac{m}{N}$ is $\frac{\text{integer}}{\text{integer}}$ is a rational number.

↳ This condition also applies for sinosidal signals.

Example

$x(t) = \cos\left(\frac{2\pi}{12} t\right)$ → Periodic $T = 12$

$x[n] = \cos\left[\frac{2\pi}{12} n\right]$ → $\frac{\omega_0}{2\pi} = \frac{1}{12} \rightarrow N$

↳ periodic at $N = 12$.

Example

$x(t) = \cos\left(\frac{8\pi}{31} t\right)$ → Periodic $T = 31/4$

$x[n] = \cos\left[\frac{8\pi}{31} n\right]$ → periodic ↓ $N = 31$

Example

$x(t) = \cos\left(\frac{t}{6}\right)$

Periodic with $T_0 = \cancel{\pi/3}\ 12\pi$

$$x[n] = \cos\left(\frac{n}{6}\right) \qquad \frac{\omega_0}{2\pi} = \frac{1/6}{2\pi} = \frac{1}{12\pi}$$

not periodic ← not rational

It is not necessary that a signal in continous time and discrete time have the same period.

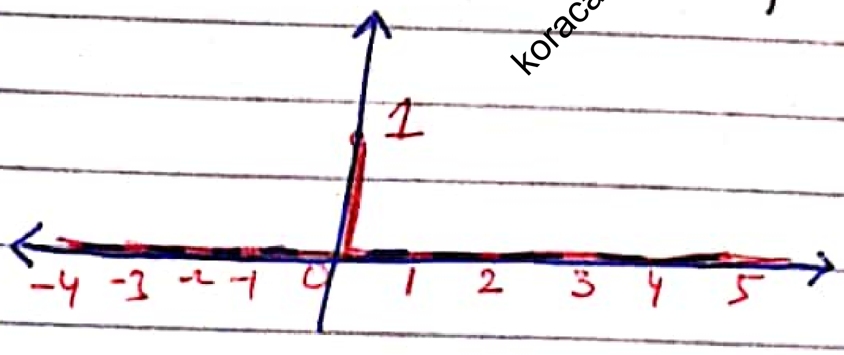
Lecture 3 03/03/20

## ② Unit Impulse and Unit Step Signals

First we consider the discrete time domain.

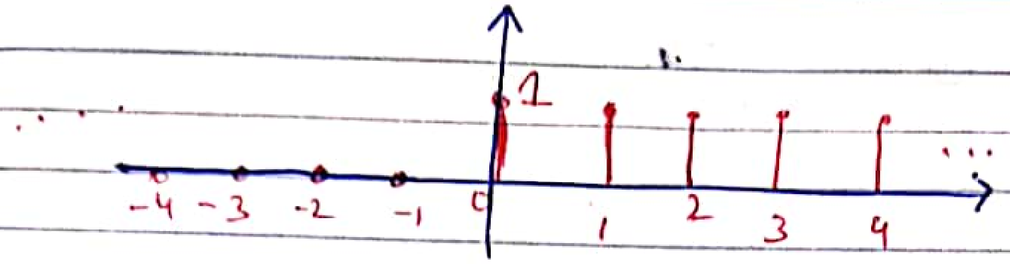
### Unit Impulse Signal:

$$\delta[n] = \begin{cases} 0 & \text{for } n \neq 0 \\ 1 & \text{for } n = 0 \end{cases}$$




### Unit step signal

$$u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$$





## Representing Unit Impulse in terms of Unit step.

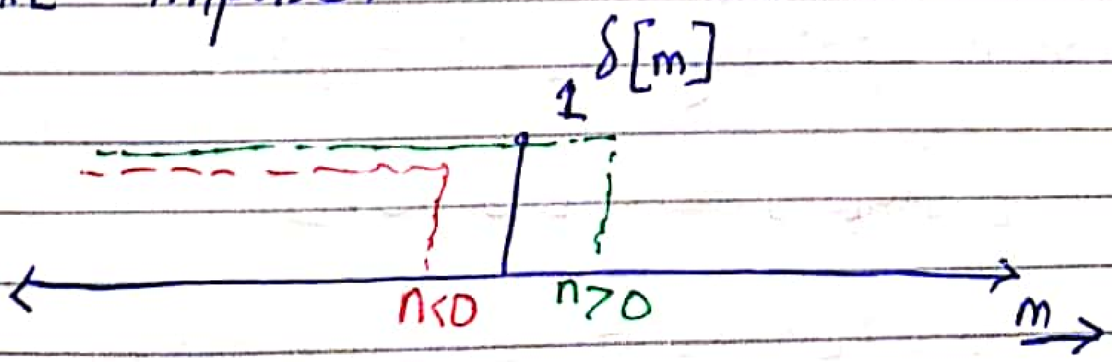
$$\delta[n] = u[n] - u[n-1]$$

Unit impulse signal is the first differente of unit step.

## Representing unit step in terms of unit impulse signal

$$u[n] = \sum_{m=-\infty} \delta[m]$$

Unit step is the running sum of unit impulse.





$$u[n] = \sum_{m=-\infty}^{n} \delta[m] = \begin{cases} 0 & n<0 \\ 1 & n \geq 0 \end{cases}$$

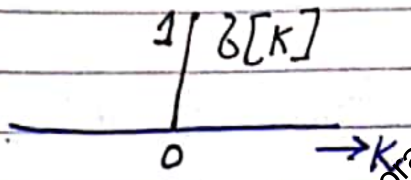
Alternate form;

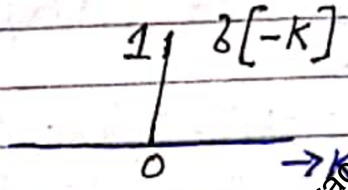
$$m \rightarrow k = n-m$$

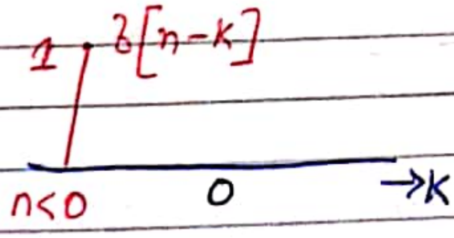
$$u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

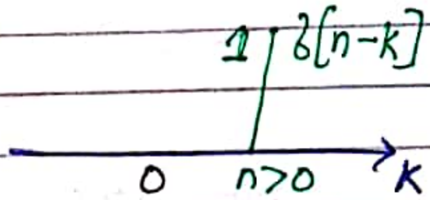
or if change limits;

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$




Sampling property of unit impulse;

$$x[n]\,\delta[n] = x[0]\,\delta[n]$$

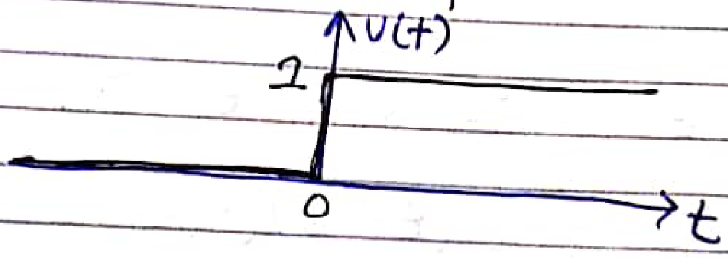
$$x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0]$$

Next we consider these signals in continous time domain.

## Unit step signal

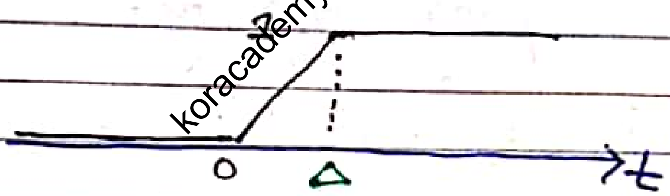
$$u(t) = \begin{cases} 0 & \text{for } t<0 \\ 1 & \text{for } t>0 \end{cases}$$

ideal



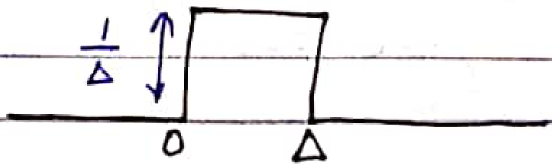

discontinuity at $t=0$

An approximated continous version;




$$u(t) = \lim_{\Delta\to 0} u_\Delta(t)$$

Unit Impulse; $\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)$

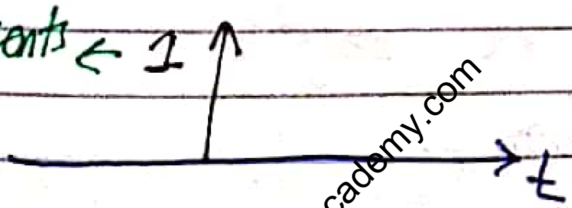



Area = 1

Ideal unit impulse;

$$\delta(t) = \lim_{\Delta\to 0} \delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)$$

this represents area ←



Unit step is the running integral of unit impulse function.

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

or let $\tau = \sigma = t - \tau$

$$u(t) = \int_{\infty}^{0} \delta(t - \sigma)(-d\sigma)$$

$$u(t) = \int_{0}^{\infty} \delta(t - \sigma)\, d\sigma$$

The sampling property;

$$x(t)\,\delta(t) = x(0)\,\delta(t)$$

$$x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

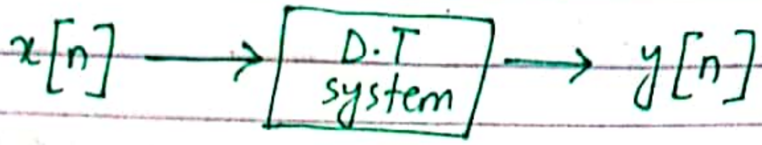
## SYSTEMS

A system is any entity that takes an input signal and provides the desired output after applying some processes.

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{C.T} \\ \text{System}\end{array}} \longrightarrow y(t)$$

$$x(t) \rightarrow y(t)$$

process C.T signals.

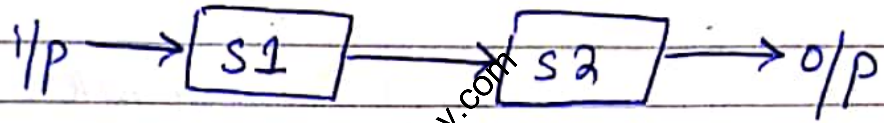



$$x[n] \to y[n]$$

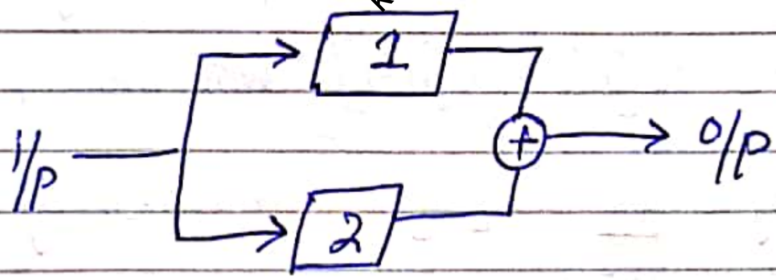
processes discrete time signals.

## Inter Connection of Systems:

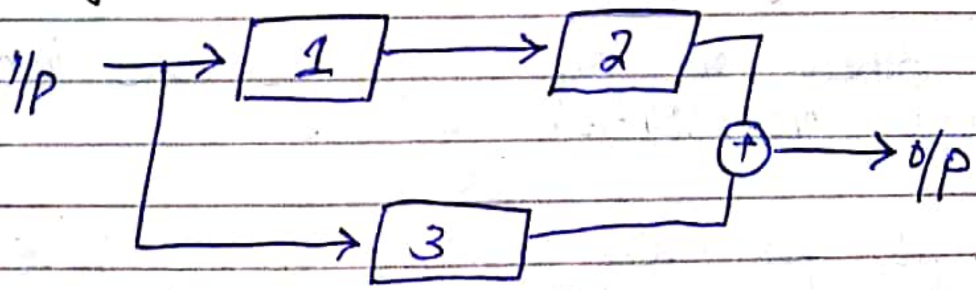
i. Series (Cascade)



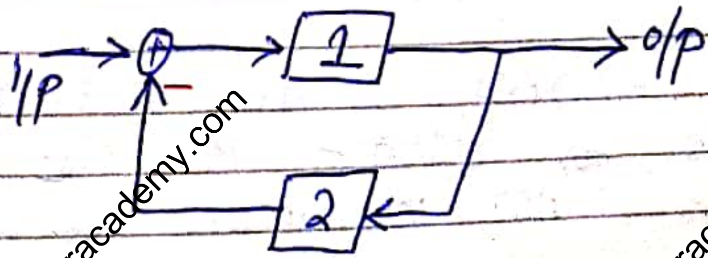

ii. Parallel




iii. Hybrid




iv. Feedback



# Basic System Properties

1) Based on memory, we have memoryless and with memory systems.

If the system o/p depends only on the present i/p is memoryless system.

If the system o/p depends on the present i/p at that time and also the past o/p is system with memory.

eg $y[n] = 2x[n]$ memoryless

$y(t) = x(t-1)$ with memory

$y[n] = x[n] + x[n-1]$ with memory

Acumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$ with memory

$y(t) = x(t+1)$ with memory

② Based on invertibility, we have invertible and non invertible systems.

If for every distinct i/p we have a distinct o/p, this system is invertible.

Whereas if we don't have distinct o/ps for distinct i/ps, the system is non invertible.

For every invertible system, an assosiated inverse system exits.

Inverse system is that system which if connected with an inverted system gives us the i/p back at o/p.

$$x(t) \longrightarrow \boxed{\text{Invt}} \xrightarrow{w(t)} \boxed{\text{Invs}} \longrightarrow x(t)$$

Example

$y(t) = 2x(t) \rightarrow$ Invertible.

$y(t) = x^2(t) \rightarrow$ non invertible

$y(t) = 0 \rightarrow NI$

Consider the accumulator;

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n]$$

$$y[n] = y[n-1] + x[n]$$

$$y[n] - y[n-1] = x[n]$$

So Invertible

as inverse system exists.

(3) Based on causality, we have causal and non causal systems.

If the o/p of system depends on present and past i/ps only is a causal system.

Whereas if it also depends on the future i/ps it is a non causal system.

Example

$$y[n] = x[n] - x[n+1] \quad \text{Non causal.}$$

$$y[n] = \sum_{K=-\infty}^{n} x[K] \quad \text{Causal}$$

$$y[n] = \frac{1}{2M+1} \sum_{K=-M}^{M} x[n-K] \quad N.C$$

Averaging system

$$y[n] = x[-n] \quad N.C$$

(4) Based on stability we have stable and unstable systems.

If we apply a bounded i/p and get a bonded o/p, such system is a stable system.

Whereas for a bounded i/p if we have an unbounded o/p, such system is unstable.

Example

Say $y(t) = t\,x(t)$ → unstable

If $x(t) = 1$

$y(t) = t$

$y(t) = e^{x(t)}$ → stable

$|x(t)| < B < \infty$

$|y(t)| < e^{B} < \infty$

⑤ Based on time invariance we have time invarient and time varying system.

If we introduce a time shift in the i/p and the o/p shifts by the same amount, so this is a time invariant system.

i.e if $x(t) \rightarrow y(t)$

so $x(t-t_0) \rightarrow y(t-t_0)$

If the i/p shift does not equal the o/p shift, then it is time varying.

Example

$y(t) = \sin|x(t)|$

For $x(t) = x_1(t)$

$y_1(t) = \sin|x_1(t)|$

$x_2(t) = x_1(t-t_0)$

$y_2(t) = \sin|x_2(t)|$

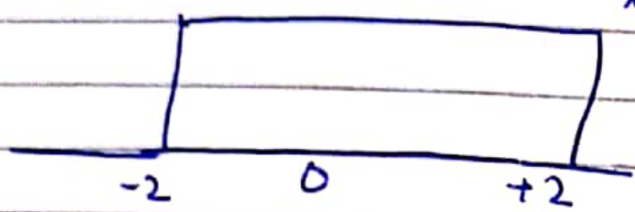
$y_2(t) = \sin|x_1(t-t_0)|$

↳ so time invariant system.

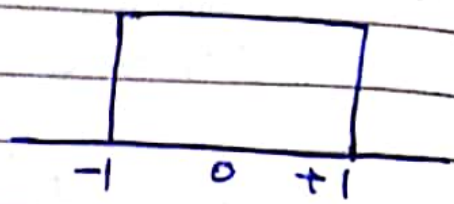
$y(t) = x(2t)$

consider

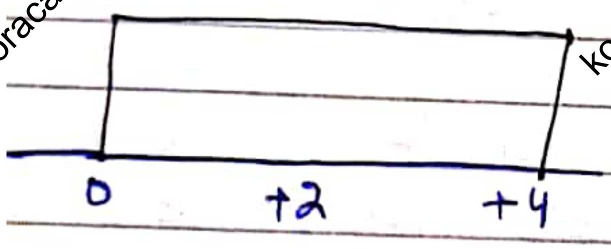
$x_1(t)$



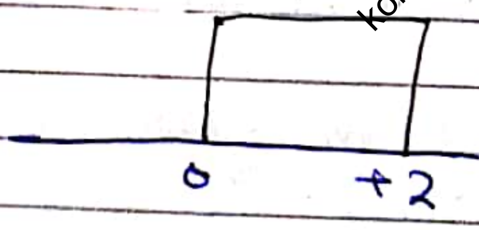

$y_1(t)$




$x_2(t) = x_1(t-2)$




$y_2(t)$





$y_1(t-2)$

$y_1(t-2) \neq y_2(t)$

↳ so time varying

# Lecture 4

10/03/20

Time invariance: System behaviour does not change with time.

$x(t) \rightarrow \boxed{\text{system}} \rightarrow y(t) \rightarrow \boxed{\text{Delay}} \rightarrow y(t-t_0)$

Similarly

$x(t) \rightarrow \boxed{\text{Delay}} \rightarrow x(t-t_0) \rightarrow \boxed{\text{system}} \rightarrow y(t-t_0)$

$y(t) \neq y(t-t_0)$

time invariant     variant.

⑥ Based on linearity, we have linear and non linear systems.

A system following the principle of superposition is a linear system.

Principle of superposition.
It has two properties;

i. Additive property.
ii. Homogenity property.

Consider;

$x_1(t) \rightarrow \boxed{\text{system}} \rightarrow y_1(t)$

$x_2(t) \rightarrow \boxed{\text{system}} \rightarrow y_2(t)$

$\oplus \rightarrow y_1(t) + y_2(t)$

Similarly if we add first and then apply to the system;

$$x_1(t) + x_2(t) \rightarrow \boxed{\text{system}} \rightarrow y'(t) = y_1(t) + y_2(t)$$

$$y'(t) \neq y_1(t) + y_2(t)$$

~~Lin~~ obeys the additive property.
does not obey.

Similarly the homogenity property;

$$x(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t) \rightarrow \boxed{K} \longrightarrow K\,y(t)$$

whereas if

$$x(t) \rightarrow \boxed{K} \xrightarrow{Kx(t)} \boxed{\text{system}} \longrightarrow y'(t) = K\,y(t)$$

$$y'(t) \neq K\,y(t)$$

obeys homogenity property
does not obey

A linear system must obey both the additive and homogenity properties.

For a linear system;

$$a x_1(t) + b x_2(t) \rightarrow a y_1(t) + b y_2(t)$$

Similarly for discrete time domain also,

$$a x_1[n] + b x_2[n] \rightarrow a y_1[n] + b y_2[n]$$

## Examples.

$$y(t) = t\,x(t) \rightarrow \text{linear system.}$$

$$y_1(t) = t\,x_1(t) \qquad y_2(t) = t\,x_2(t)$$

$$y_1(t) + y_2(t) = t\,[x_1(t) + x_2(t)]$$

when i/p $x_1(t) + x_2(t)$

so $y'(t) = t\,[x_1(t) + x_2(t)]$.

So additive property holds and also the homogenity property does.

$$y(t) = x^2(t) \rightarrow \text{Non linear.}$$

$$y[n] = 2x[n] + 3 \rightarrow \text{non linear.}$$

For a linear system, if the i/p is zero the o/p must also be zero.

## Chapter 2

## LTI Systems

LTI → linear time invariant systems.

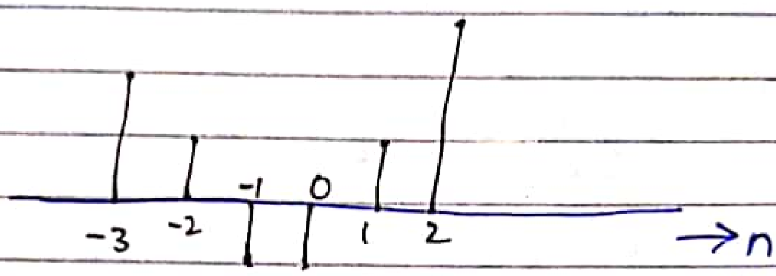
$$x(t) \rightarrow \boxed{LTI} \rightarrow y(t)$$

Considering the discrete time domain first;

## Discrete time LTI system

Representing a given signal in terms of a basic signal i.e delta signal $\delta[n]$.

Consider $x[n]$ as shown.

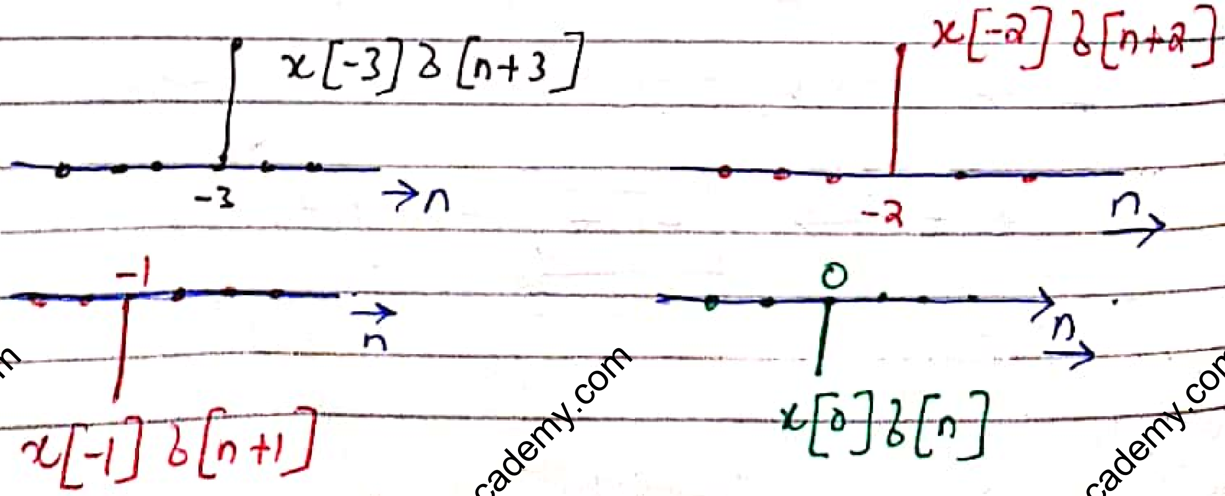



$\rightarrow x[-3]\,\delta[n+3] = \begin{cases} x[-3] & \text{for } n = -3 \\ 0 & \text{for } n \neq -3 \end{cases}$

$\rightarrow x[-2]\,\delta[n+2] = \begin{cases} x[-2] & n = -2 \\ 0 & n \neq -2 \end{cases}$

$\rightarrow x[-1]\,\delta[n+1] = \begin{cases} x[-1] & n = -1 \\ 0 & n \neq -1 \end{cases}$

$\rightarrow x[0]\,\delta[n] = \begin{cases} x[0] & , n = 0 \\ 0 & , n \neq 0 \end{cases}$



So this $x[n] = x[-3]\,\delta[n+3] + x[-2]\,\delta[n+2]$
$+ x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1]$
$+ x[2]\,\delta[n-2]$.

So generalizing for any signal $x[n]$;

$$x[n] = \sum_{K=-\infty}^{+\infty} x[K]\,\delta[n-K].$$

<u>Discrete time LTI system unit impulse response and convolution sum</u>

$$x[n] = \delta[n] \longrightarrow \boxed{\text{LTI system}} \longrightarrow y[n] = \underbrace{h[n]}_{\text{unit impulse response.}}$$

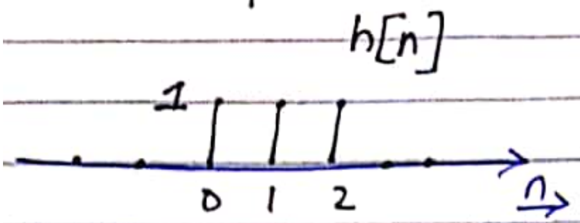
$$x[n] = \delta[n-\overset{\text{delay}}{K}] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = h[n-K].$$

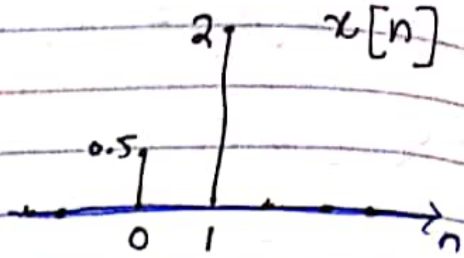
$$x[n] = x[k]\,\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = x[k]\,h[n-k]$$

$$x[n] = \sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K] \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = \sum_{K=-\infty}^{\infty} x[k]\,h[n-K]$$

Convolution sum. $= x[n] * h[n]$.
Convolution operator.

# Example




$$y[n] = ?$$

$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$$

As it is non zero only at $n = 0$ and $1$.

So $y[n] = x[0]\,h[n-0] + x[1]\,h[n-1]$

$\Rightarrow y[n] = 0.5\,h[n] + 2\,h[n-1] \rightarrow ①$

To plot $y[n]$, we use ①

$h[n] = 0 \quad h[n-1] = 0$

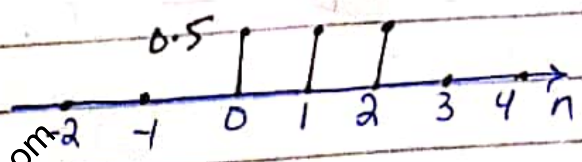
$y[-1] = 0$

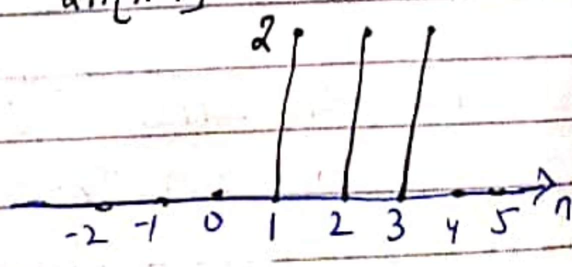
$y[n] = 0$ for $n < 0$

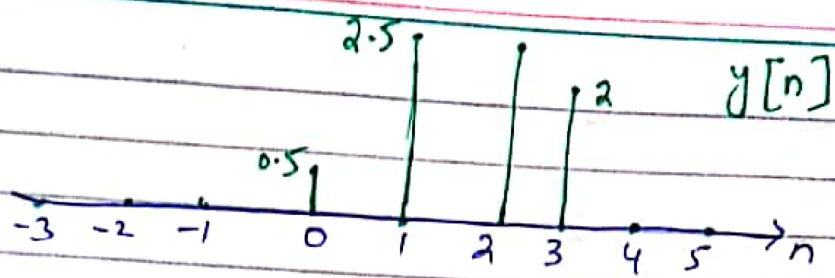
$y[0] = 0.5 \qquad y[1] = 2.5 \qquad y[2] = 2.5$

$y[3] = 2 \qquad y[4] = 0 \qquad y[n] = 0$ for $n \geq 4$

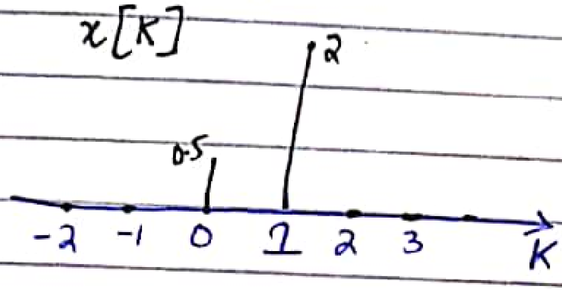
① $0.5\,h[n]$ $\qquad 2h[n-1]$

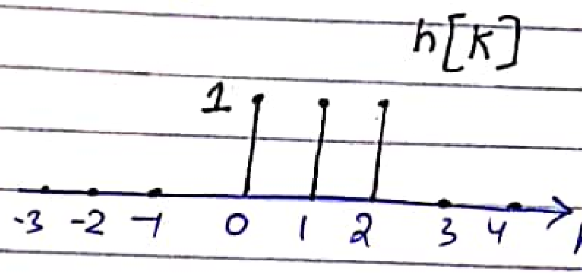


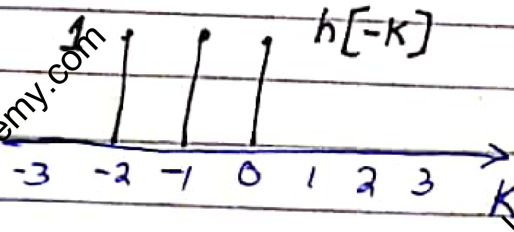



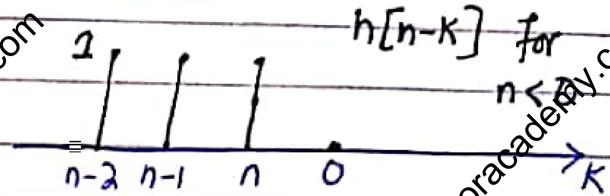



The same example by another method;
As $x[n]$ and $h[n]$ are given.




$$y[n] = \sum_{K=-\infty}^{\infty} x[K]h[n-K]$$

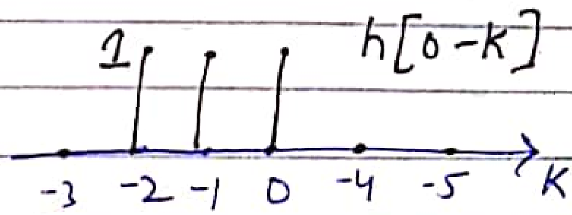
For $n<0$

$$y[n]=0$$

As product of $x[K]$ and $h[n-K]$ is zero for $n<0$.

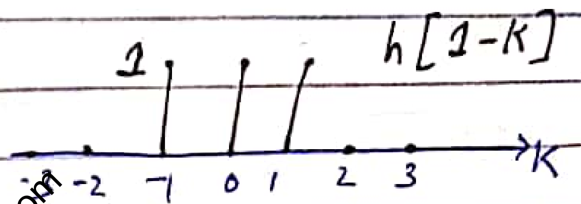
For $n=0$

$$y[0] = 0.5$$




For $n=1$

$$y[1] = 2.5$$



## Convolution sum

$$x[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} \alpha^n & 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases} \qquad \alpha > 1$$

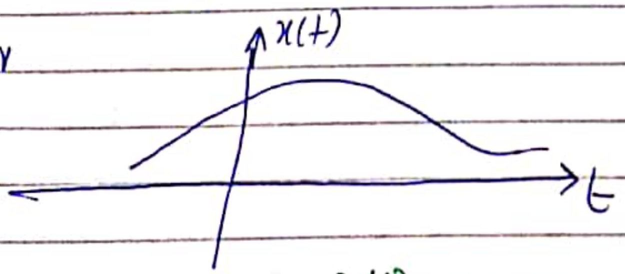
$y[n] = ?$ $\qquad y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$

## Lecture 5 02/06/20

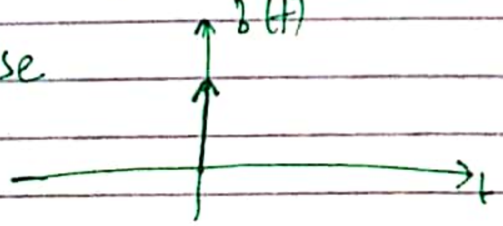
## Continous Time LTI System

The same strategy i.e represent any CT signal $x(t)$ as a linear combination of basic signals (unit impulse).

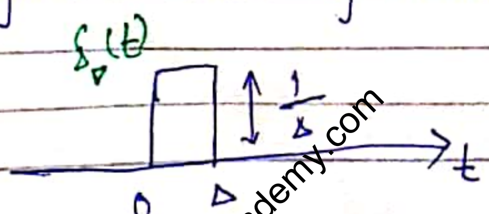
Consider




unit impulse




This signal is basically derived from;



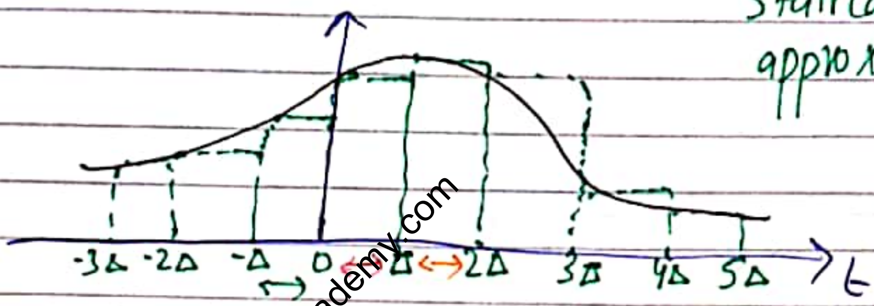
width $\Delta$ and height $1/\Delta$. Area = 1

So decreasing width increases height.

$$\lim_{\Delta \to 0} \delta_\Delta(t) \to \delta(t)$$

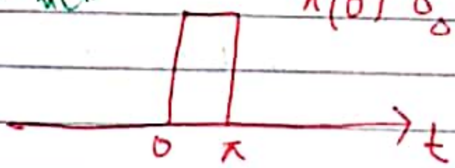
So the given signal $x(t)$

staircase approximation



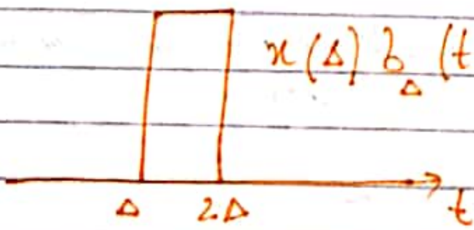

to set value to $x(t)$

$x(0)\, \delta_\Delta(t)\, \Delta = x(0)$

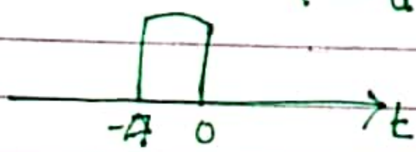



$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & \text{otherwise} \end{cases}$$

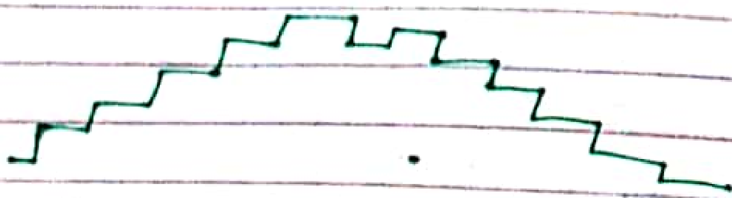
$x(\Delta)\, \delta_\Delta(t-\Delta)\, \Delta$




$x(-\Delta)\, \delta(t+\Delta)\, \Delta$



The staircase approximated signal is as;



$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\, \delta_\Delta(t - k\Delta)\, \Delta$$

How to make this approximation more accurate to $x(t)$?

↳ Apply limit to $\Delta$.

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\, \delta_\Delta(t - k\Delta) \cdot \Delta$$

Summation to integration. when duration is very small, DT to CT.

$\delta_\Delta$ becomes ideal impulse ($\delta$)

$k\Delta$ to $\tau$ → b/c $k\Delta$ is a discrete number and we we need a continuous.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t - \tau)\, d\tau$$

## Unit Impulse Response

$$\hat{x}(t) = \delta_\Delta(t) \rightarrow \boxed{\text{LTI system}} \rightarrow y(t) = h_\Delta(t)$$

Time invariance property:

$$\delta_\Delta(t-k\Delta) \rightarrow \boxed{LTI} \rightarrow h_\Delta(t-k\Delta)$$

Homogenity property

$$x(k\Delta)\,\delta_\Delta(t-k\Delta) \rightarrow \boxed{LTI} \rightarrow x(k\Delta)\,h_\Delta(t-k\Delta)$$

A signal applied will consist of a number of impulses

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta \rightarrow \boxed{LTI}$$

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,h_\Delta(t-k\Delta)\,\Delta$$

Again if $\Delta$ is very small
$\Delta \rightarrow 0$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \rightarrow \boxed{LTI}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

unit impulse response

convolution ~~sum~~ integral

$$y(t) = x(t) * h(t)$$

↳ convolution operator.

# Lecture 6 09/06/20

## DT LTI system

$x[n] = \alpha^n u[n]$ $\quad 0 < \alpha < 1$

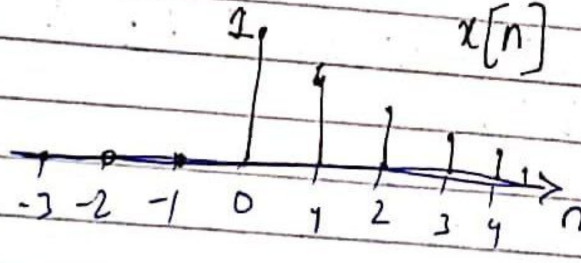
$h[n] = u[n]$

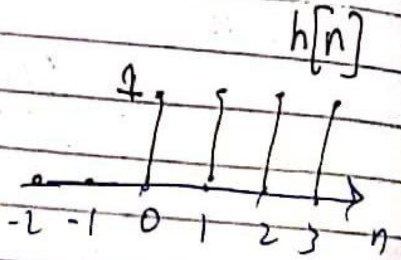
$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = ?$

As $y[n] = x[n] * h[n]$

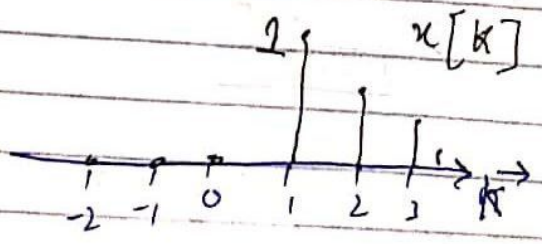
$$= \sum_{-\infty}^{\infty} x[k]\, h[n-k]$$

Solving graphically.





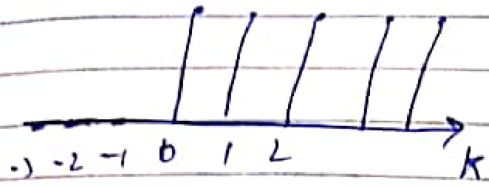
For $x[k]$ only replace variables i.e. n by k.



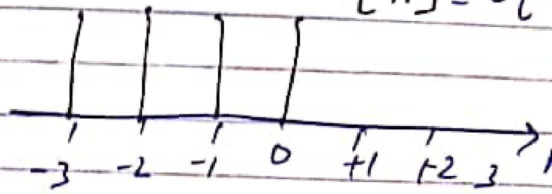
∴ We need to draw $h[n-k]$ for a particular value of n.

↳ we are interested in all possible values of n.

$h[K] = u[K]$



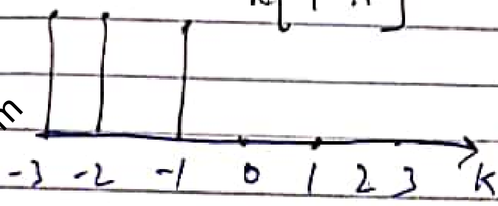
$h[-K] = u[-K]$



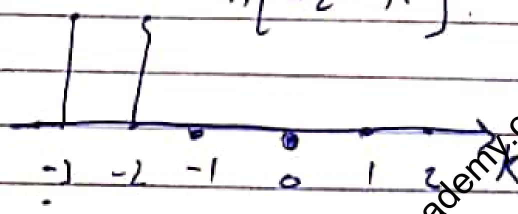
Now $h[n-K]$

Say for a -ve value ie $-1$ ie $h[-1-K]$ the signal $h[-K]$ will be shifted left by 1 unit.

$h[-1-K]$



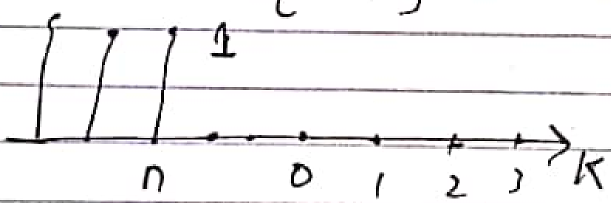
$h[-2-K]$



So;

$x[n-K]$

$n<0$



So the product; $x[K]\,h[n-K] = ?\ 0$ for $n<0$.

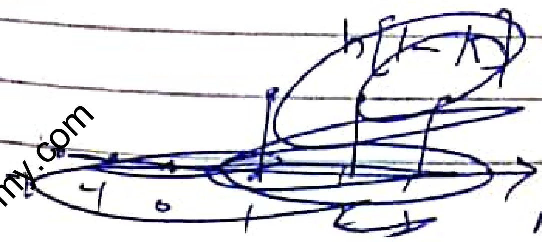
as no overlap.

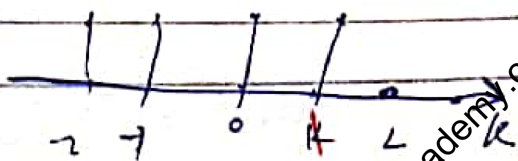
So therefore $y[n] = 0$ for $n<0$.

Similarly for positive values of $n$.

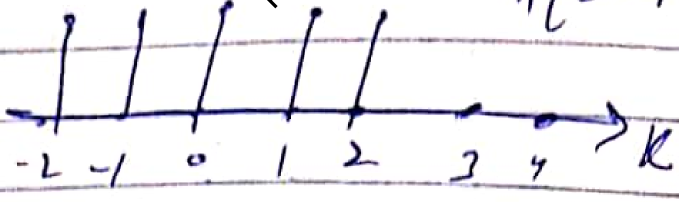
We plot $h[n-K]$ for $n>0$.

$h[1-K]$

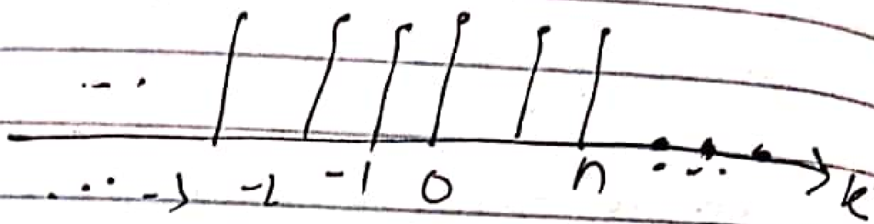




$h[2-k]$




Similarly $\qquad h[n-k] \qquad n>0$




The product;

$$x[k]\, h[n-k] = \begin{cases} \alpha^{k} & , 0 \le k \le n \\ 0 & , \text{otherwise} \end{cases}$$

So for $n \geq 0$

$$y[n] = \sum_{k=0}^{n} \alpha^{k} = \frac{1-\alpha^{n+1}}{1-\alpha}$$

So the final answer; ie combining for $n>0$ and $n<0$

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right] u[n]$$

Ans

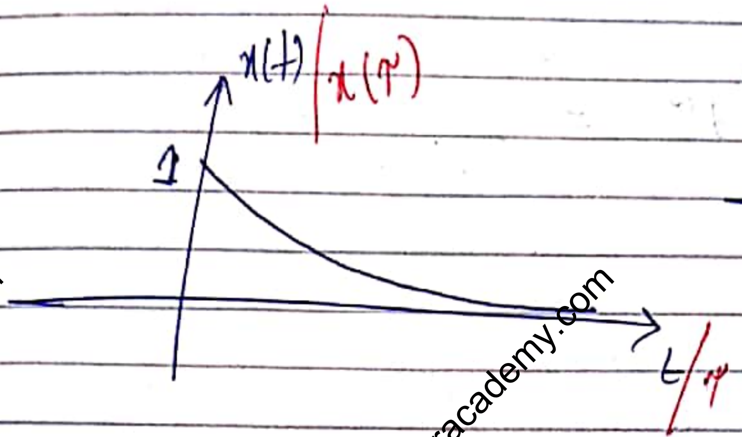
for all values of $n$.

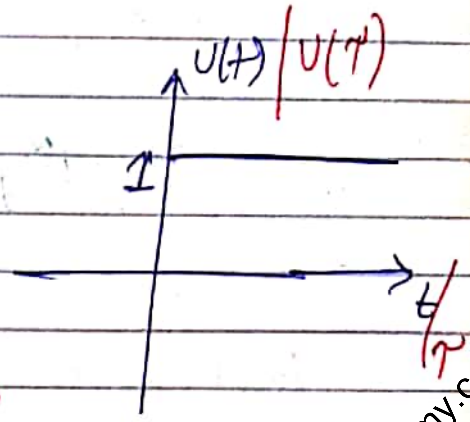
**CT** $x(t) = e^{-at} u(t)$ $a>0$

$h(t) = u(t)$

$y(t) = ?$

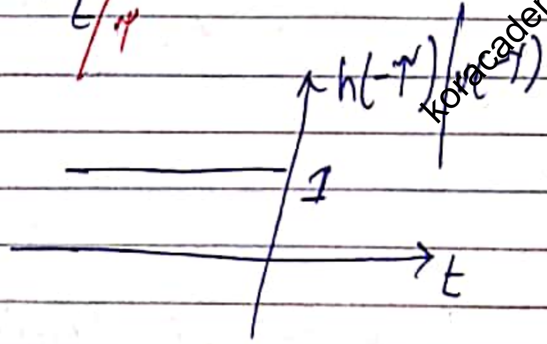
As $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau =$



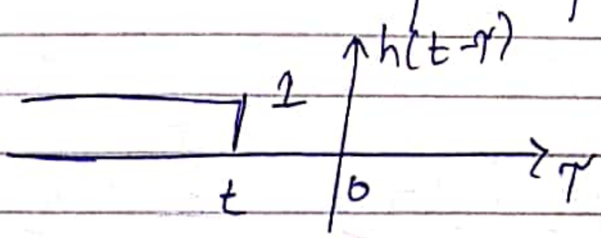




$h(t-\tau) = ?$

$t \rightarrow +ve$, $-ve$




If $t = -ve$
$h(t-\tau)$ means $h(-\tau)$ shifted left.




for $t<0$
$x(\tau)\, h(t-\tau) = 0$ $\Rightarrow y(t) = 0$

If $t>0$
$h(t-\tau)$ means $h(-\tau)$ shifted right.


So for $t>0$

$$x(\tau)\,h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 \le \tau \le t \\ 0, & \text{otherwise} \end{cases}$$

$$y(t) = \int_0^t e^{-a\tau}\,d\tau = \frac{1}{a}\left(1-e^{-at}\right)$$

Final answer:

$$y(t) = \frac{1}{a}\left(1-e^{-at}\right)u(t)$$

↳ why $u(t)$?

B/c $y(t)=0$ for $t<0$.

Properties of Convolution operator

i. The commutative property. ie

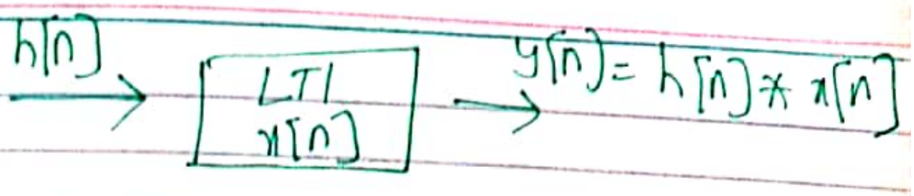
$$x[n] * h[n] = h[n] * x[n]$$

$$\sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

left

$$x[n] \longrightarrow \boxed{\begin{matrix}LTI \\ h[n]\end{matrix}} \longrightarrow y[n] = x[n] * h[n]$$

Right

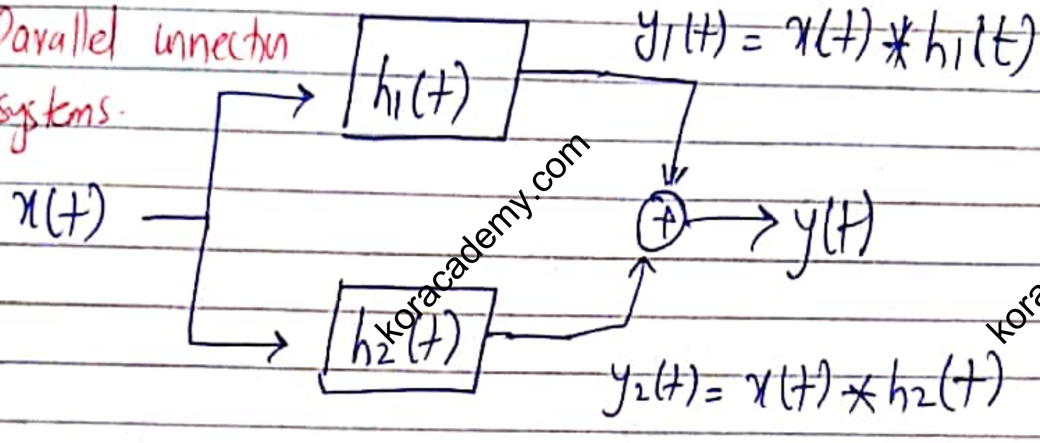



ii. The Distributive property.

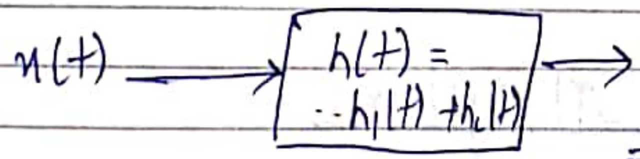
$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

Similarly for discrete time.

Parallel connection of systems.



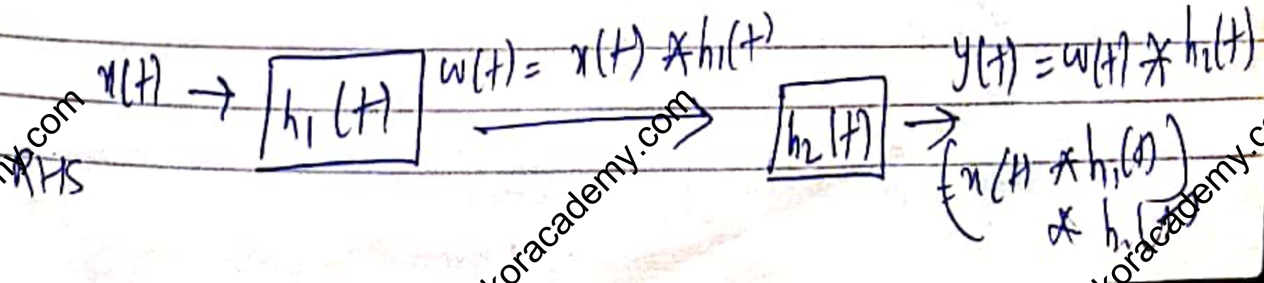

single system




$$y(t) = x(t) * h(t)$$
$$= x(t) * [h_1(t) + h_2(t)]$$

iii. The assosiative Property.

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$



LHS

$$x(t) \longrightarrow \boxed{h(t) = h_1(t) * h_2(t)} \rightarrow y(t) = x(t) * \left(h_1 * h_2(t)\right)$$

Using commutative property;

$$x(t) \longrightarrow \boxed{h_2(t) * h_1(t)} \longrightarrow y(t) = x(t) * \left(h_2 * h_1\right)$$

Similarly for two cascaded systems;

$$x(t) \rightarrow \boxed{h_2(t)} \longrightarrow \boxed{h_1(t)} \rightarrow y(t) =$$

The order in which LTI systems are connected in series doesn't matter.

$$x(t) - \boxed{1} - \boxed{2} - \boxed{3} - \boxed{4} - y(t)$$

As

$$x(t) - \boxed{4} - \boxed{1} - \boxed{2} - \boxed{3} - y(t)$$

## Properties of LTI systems.

System behaviour is completely characterized by its unit impulse response.

### ① LTI system's memory

As $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

K can vary from $-\infty$ to $\infty$ → present, past, future

For memoryless; present i/p; $k = n$

$k \neq n$ → past, → future  $k < n$ past, $k > n$ future

For LTI system without memory;

$$\boxed{h[n] = 0 \quad \text{for } n \neq 0; \quad \Rightarrow h[n] = k\,\delta[n]}$$ , k is some constant

For with memory; $\boxed{h[n] \neq k\,\delta[n]}$

CT;

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

without memory; $h(t) = 0$, for $t \neq 0$

$h(t) = k\,\delta(t)$.

with memory; $h(t) \neq k\,\delta(t)$

without memory $h[n] = k\,\delta[n]$,

$$y(n) = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$\boxed{y[n] = k\,x[n]}$$

Similarly CT; $y(t) = k\,x(t)$.

$t - \tau$ ;  $-\infty \le \tau \le \infty$

$\tau < t$ past

$\tau > t$ ft

✓ $\tau = t$ = prst

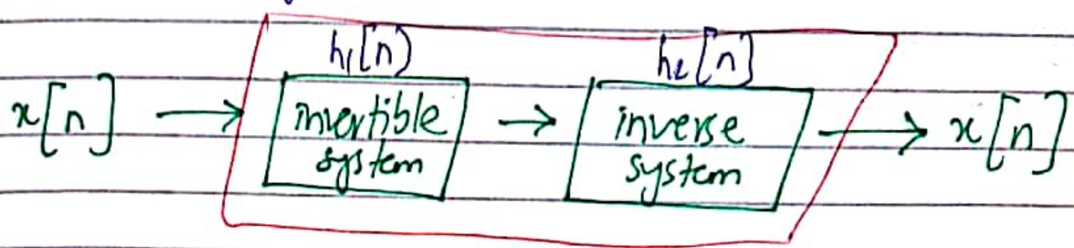
$\tau \neq t$ ; $t - \tau \neq 0 \rightarrow h = 0$

16/06/2020

## ② LTI Systems Invertibility

For invertible system → unique input ⇒ unique output.

An invertible system will always have an inverse system.




The overall behaviour of the system is an identity system.

So its impulse response $h[n] = \delta[n] \rightarrow (A)$

$$h[n] = h_1[n] * h_2[n] \rightarrow (B)$$

(A) and (B) $\Rightarrow$ $h_1[n] * h_2[n] = \delta[n]$

Similarly for continuous time LTI systems;

$$\underset{\text{Invrtble}}{h_1(t)} * \underset{\text{Invrs}}{h_2(t)} = \delta(t)$$

Example

$$y(t) = x(t - t_0)$$

Is it invertible?

$$h_1(t) = \delta(t - t_0)$$

If $h_1(t) * h_2(t) = \delta(t)$
then This system is invertible

& $\delta(t-t_0) * \delta(t+t_0) = \delta(t)$

$\delta(t) = \delta(t)$

## ③ LTI system Causality

**DT LTI system**

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$y[n]$ ↓ o/p

$x[k]$ ↓ i/p

$h[n-k]$ ↓ unit impulse response

If $k < n$ past
$k = n$ present
$k > n$ future

For causal LTI system;

$$h[n] = 0 \quad \text{for } n < 0$$

**For CT LTI system**

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

For causal value we don't want future value

$h(n) = \delta(n) + \delta(n-1)$
$h(n) = 0$ for $n < 0$
non causal

If $\tau > t$ $x(\tau)$ is future value.
So we want impulse response to be zero.
If ~~too~~, $\tau > t$
$t - \tau < 0$
For causal LTI system;
$$h(t) = 0 \quad \text{for} \quad t < 0$$

Example
Say $h[n] = u[n]$
is it causal ?
$h[n] = 0$ for $n < 0$ so causal.

Condition of Initial Rest
If the i/p to a causal system is zero upto some point in time, then the o/p must also be 0 upto that time.
→ equivalent to causal LTI system.

Example $y[n] = x[n] + 3$
This is a non linear, causal system
Does not satisfy condition of initial rest.
As if $x[n] = 0$ $y[n] = 3 \neq 0$

## (9) LTI system Stability

System is stable if we have a bounded o/p for bounded i/p (BIBO).

### DT LTI system

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

For $|x[n]| < B$ for all $n$.
└ any finite number.

Whether $|y[n]| < \infty$ ?

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \right|$$

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} \left| x[k]\, h[n-k] \right|$$

$$|y[n]| \leq B \sum_{k=-\infty}^{\infty} \left| h[n-k] \right|$$

condition;

$$\sum_{k=-\infty}^{\infty} \left| h[n-k] \right| < \infty \quad \text{so that } y[n] < \infty$$

↓
w7 [illegible]

$\sum_{k=-\infty}^{\infty} |h[n]| < \infty \Rightarrow$ LTI system will be stable if the unit impulse response is absolutely summable

Ex

(1) if $h[n] = \delta[n-n_0]$.

Stability check?

$$\sum_{k=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n-n_0]| = 1 < \infty$$

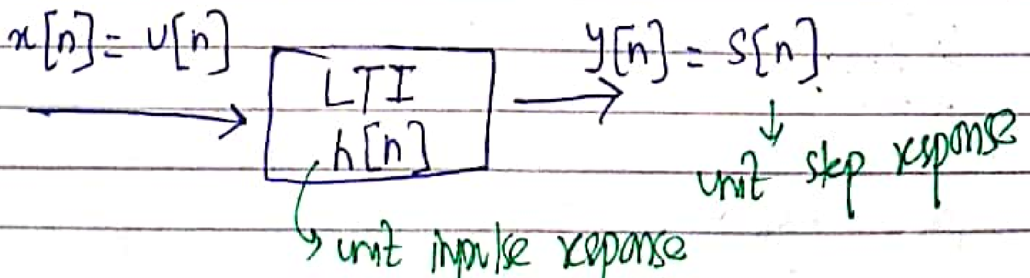
$\hookrightarrow$ stable

only exist at $n_0$ and value is 1.

(2) $h[n] = u[n]$

$$\sum_{k=-\infty}^{\infty} |h[n]| = \sum |u[n]| = \infty \rightarrow \text{unstable}$$

## Unit Step Response Of LTI systems.




$$s[n] = u[n] * h[n]$$

As convolution is commutative

$$s[n] = h[n] * u[n] \rightarrow (6)$$

$$h[n] \longrightarrow \boxed{u[n]} \longrightarrow s[n]$$

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]\, u[n-k]$$

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

A unit step response of LTI system is running sum of its unit impulse response.

$$h[n] = s[n] - s[n-1]$$

Unit impulse response is the first difference of unit step response.

# Systems described by Differential Equations.

$$x(t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow y(t)$$

Out of the many types of D-E, the eq that we use for LTI system are;

## Linear Constant Coefficient D.E (LCCDE)

For representing i/p / o/p relationship.

Whenever any system is represented by D-E, it is called implicit relationship.

We always want explicit relation.

i.e. $y(t) =$ given expression, a fn of t.

$\frac{dy(t)}{dt} - 2y(t) = x(t)$

To get explicit, we need to solve the D.E.

after solution, this will always contain a constant → so diff constant = diff soln.

but we need to find a particular eq for a particular system → so we need additional information → initial condition.

auxiliary condition

For Causal LTI system described by LCCDE we used the condition of initial rest as auxiliary condition.

**Ex.**

$$\frac{d}{dt}y(t) + 2y(t) = x(t) \rightarrow (A)$$

$$x(t) \longrightarrow \boxed{sys} \longrightarrow y(t)$$

For $x(t) = K e^{3t} \cdot u(t)$ $\quad y(t) = ?$

Sol $\quad y(t) = y_p(t) + y_h(t)$

particular $\quad$ homogenous

For particular i/p $\quad$ solution of homogenous equation

In the given eq, when $x(t) = 0$ = homogenous eq → solution

$$x(t) = Ke^{3t} \quad ; \quad t > 0$$

Assume $\quad y_p(t) = Y e^{3t}$ satisfy A.

Substitute this in (A) → [illegible]

(A) ⇒ $3Ye^{3t} + 2Ye^{3t} = Ke^{3t}$

$$\Rightarrow \boxed{Y = \frac{K}{5}}$$

$$\Rightarrow \boxed{y_p(t) = \frac{K}{5} e^{3t}\, u(t)}$$

Now $y_h(t) = ?$

$\Rightarrow \quad \frac{dy(t)}{dt} + 2y(t) = 0 \quad \rightarrow$ (B)

Assume $y_h(t) = Ae^{st}$ is the solution

(B) $\Rightarrow \quad Ase^{st} + 2Ae^{st} = 0$

$Ae^{st}(s+2) = 0$

Either $A=0$ which contradicts
so condition is satisfied if $s=-2$

$\Rightarrow \quad y_h(t) = Ae^{-2t}$

Complete solution;

$$\boxed{y(t) = \frac{K}{5}e^{3t} + Ae^{-2t}}, \quad t \geq 0$$

we still need to find $A$.

So auxiliary condition;

causal will always fulfill condition of initial rest.

$x(t) = 0$ for $t < 0$.

$\Rightarrow y(t) = 0$ for $t < 0$

$\Rightarrow$ At $t=0$, $x(t)=0 \Rightarrow y(t) = 0$

$\Rightarrow 0 = \frac{K}{5} + A \Rightarrow \boxed{A = -\frac{K}{5}}$

$\Rightarrow y(t) = \frac{K}{5} e^{2t} - \frac{K}{5} e^{-2t}$

$\Rightarrow y(t) = \frac{K}{5}\left(e^{2t} - e^{-2t}\right)$, for $t>0$

so for all times:

$\boxed{y(t) = \frac{K}{5}\left(e^{2t} - e^{-2t}\right).u(t)}$ for all time

We solved a first order LCCDE, but this discussion is valid for any order DE

## LTI systems described by LCCDE.

$x(t) \rightarrow \boxed{LTI} \rightarrow y(t)$

i/p o/p relation may be described by such an eq.

$\frac{dy(t)}{dt} + 2y(t) = x(t)$

↳ LCCDE. → $1^{st}$ order

→ Implicit representation

↳ not possible to describe system properly.

Normally, we are interested in explicit.

## Higher order DE (LCC)

Generally $n^{th}$ order LCCDE can be represented by:

$$\boxed{\sum_{K=0}^{N} a_K \frac{d^K y(t)}{dt^K} = \sum_{K=0}^{M} b_K \frac{d^K x(t)}{dt^K}}$$

↳ Implicit rep.

? How to find explicit representation?

1. Find solution of LCCDE.
2. Solution i.e $y(t) = y_p(t) + y_h(t)$

particular solution (for particular i/p)

Homogeneous solution ($x(t) = 0$)

Homogeneous equation $\leftarrow \sum a_k \frac{d^k y(t)}{dt^k} = 0$

3. In the solution, we will have unknown constant (infinite solutions)
we are interested in finding behaviour of the system and that behaviour will be one behaviour
one particular value of constant.
4. To find unique value of constant, we require auxilary condition.

5. For causal LTI systems, the auxilary condition is always the condition of initial rest.

number of unknown constant = order of equation

If $x(t) = 0$ for $t \le t_0$; $y(t) = 0, t \le t_0$

**Auxilary conditions for $n^{th}$ order LCCDE.**

$x(t_0) = 0 \Rightarrow y(t_0) = 0 \rightarrow 1^{st}$ condition

other possibilities;

$\frac{d}{dt} y(t_0) = 0$, $\frac{d^2}{dt^2} y(t_0) = 0$, ----

Generalizing;

$$\boxed{y(t_0) = \frac{d}{dt} y(t_0) = \frac{d^2}{dt_2} y(t_0) = \cdots\cdots = \frac{d^{N-1}}{dt_{N-1}} y(t_0) = 0}$$

'N' number of auxilary conditions.

## DT LTI Systems described by Linear constant coefficient Difference Equation.

Ex.

$$x[n] \longrightarrow \boxed{\text{DT LTI}} \longrightarrow y[n]$$

let i/p o/p relationship;

$$y[n] + 2y[n-1] = x[n]$$

1st order LCCDE; 1st difference

Generalizing; ie $N^{th}$ order LCCDE;

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]}$$

Solution aspect;

↳ $y[n] = y_p[n] + y_h[n]$

If $N=0$;

$$a_0 y[n] = \sum_{k=0}^{M} b_k x[n-k].$$

Non recursive equation ⇒ $\boxed{y[n] = \frac{1}{a_0} \sum_{k=0}^{M} b_k x[n-k]}$ does not require any $y[n-k]$ only $x$.

↳ does not require any auxilary conditions.

Non recursive system.

## For $N \geq 1$.

$$\sum_{K=0}^{N} a_K y[n-K] = \sum_{K=0}^{M} b_K x[n-K].$$

$$\Rightarrow a_0 y[n] + \sum_{K=1}^{N} a_K y[n-K] = \sum_{K=0}^{M} b_K x[n-K]$$

Rearranging in terms of $y[n]$;

$$\boxed{y[n] = \frac{1}{a_0}\left\{\sum_{K=0}^{M} b_K x[n-K] - \sum_{K=1}^{N} a_K y[n-K]\right\}}$$

Will be able to find $y(n)$ if we know $y[n-K]$.

→ Recursive equation → Recursive system.

We get $y[n-K]$ from auxiliary conditions.

Recursive system o/p depends on previous system o/p.

**Non recursive Systems.**

$$y[n] = \frac{1}{a_0}\sum_{K=0}^{M} b_K x[n-K]$$

Bringing $a_0$ into summation;

$$y[n] = \sum_{K=0}^{M} \frac{b_K}{a_0} x[n-K] \quad \text{— (A)}$$

Consider; $x[n] \rightarrow \boxed{\begin{matrix}LTI \\ h[n]\end{matrix}} \rightarrow y[n]$

$$\Rightarrow y[n] = x[n] * h[n]$$

$$\text{or } y[n] = h[n] * x[n]$$

Using

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \quad \text{—} (B)$$

Both (A) and (B) are representing behav of LTI systems.

Comparing $\Rightarrow h[k] = \begin{cases} b_k/a_0 , & 0 \le k \le M \\ 0 , & \text{otherwise.} \end{cases}$

$$\Rightarrow \boxed{h[n] = \begin{cases} b_n/a_0 , & 0 \le n \le M \\ 0 , & \text{otherwise} \end{cases}}$$

→ Unit Impulse response of non recursive systems.

→ It has a finite duration.

→ FIR systems. (Finite Impulse response).

All non recursive systems are FIR systems.

## Auxilary Condition.

DT LTI system is also causal, so it must satisfy condition of initial rest.

ie. If $x[n] = 0$ for $n < n_0$

$y[n] = 0$ for $n < n_0$

$1^{st}$ order $\rightarrow$ $y[-1]$, $2^{nd}$ $\rightarrow$ $y[-1], y[-2]$

These all times are before $n_0$.

$$\boxed{y[-N], y[-N+1], y[-N+2] \cdots, y[-1] = 0}$$

Ex $y[n] - \frac{1}{2} y[n-1] = x[n]$

This is a Recursive system b/c to find $y[n]$, we need to know $y[n-1]$.

Solving $y[n] = x[n] + \frac{1}{2} y[n-1]$

For $x[n] = k\,\delta[n]$

As causal sys $\rightarrow$ so un of mi rest;

ie $x[n] = 0$ for $n \leq -1$

$\Rightarrow y[n] = 0$ for $n \leq -1$

$y[n] = ?$ for $n \geq 0$

As $y[n] = x[n] + \frac{1}{2} y[n-1]$

$\underline{n=0}$

$y[0] = x[0] + \frac{1}{2} y[-1] = \underline{K}$

$\underline{n=1}$

$y[1] = x[1] + \frac{1}{2} y[0] = \frac{1}{2} K$

$\underline{n=2}$

$y[2] = x[2] + \frac{1}{2} y[1] = \frac{1}{4} K = \left(\frac{1}{2}\right)^2 K$

Generally, for any value, $\boxed{y[n] = \left(\frac{1}{2}\right)^n K}$

$$x[n] = k\delta[n] \rightarrow \boxed{LTI} \rightarrow y[n] = \left(\frac{1}{2}\right)^n k$$

If $k = 1$;

$$x[n] = \delta[n] \rightarrow \boxed{LTI} \rightarrow y[n] = h[n] = \left(\frac{1}{2}\right)^n$$

$$\Rightarrow \boxed{h[n] = \left(\frac{1}{2}\right)^n u[n]}$$

→ The unit impulse response of recursive system has infinite duration.

→ IIR → infinite impulse response.

Block Diagram representation of systems described by Differential/Difference equation.

solving = information.

Graphical representation = visible = more informative
→ another perspective about system behavior.

Basic building blocks.

1. Adder

$$x_1 \rightarrow \underset{\uparrow x_2}{\oplus} \rightarrow x_1 + x_2$$

2. Multiplier $x \xrightarrow{a} ax$

Unit

3. <u>Delay</u>.

$$x[n] \longrightarrow \boxed{D} \longrightarrow x[n-1]$$

This is in case of DT systems.
In CT, we will sometimes require a differentiator.

$$x(t) \longrightarrow \boxed{D} \longrightarrow \frac{d}{dt}x(t)$$

4. <u>Integrator</u>:

$$x(t) \longrightarrow \boxed{\int} \longrightarrow \int_{-\infty}^{t} x(\tau)\,d\tau$$

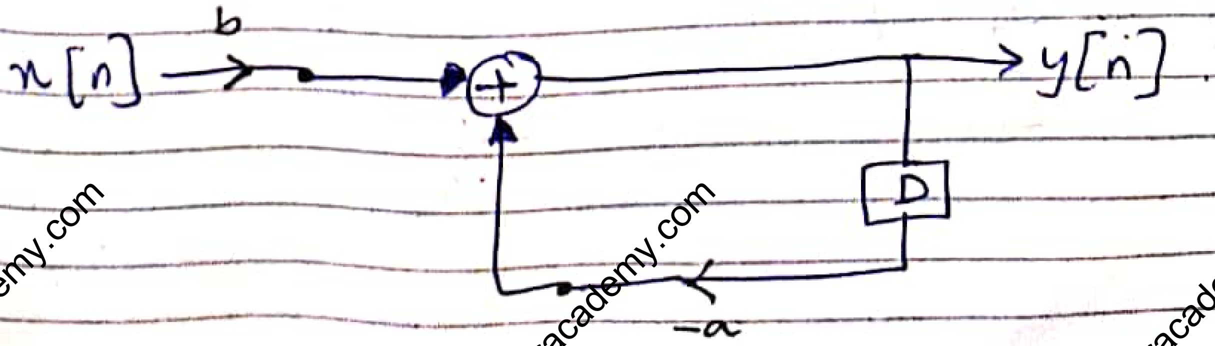
<u>Example</u>

$$y[n] + a\,y[n-1] = b\,x[n]$$

LTI, 1st order, LCCDE
Block diagram ?

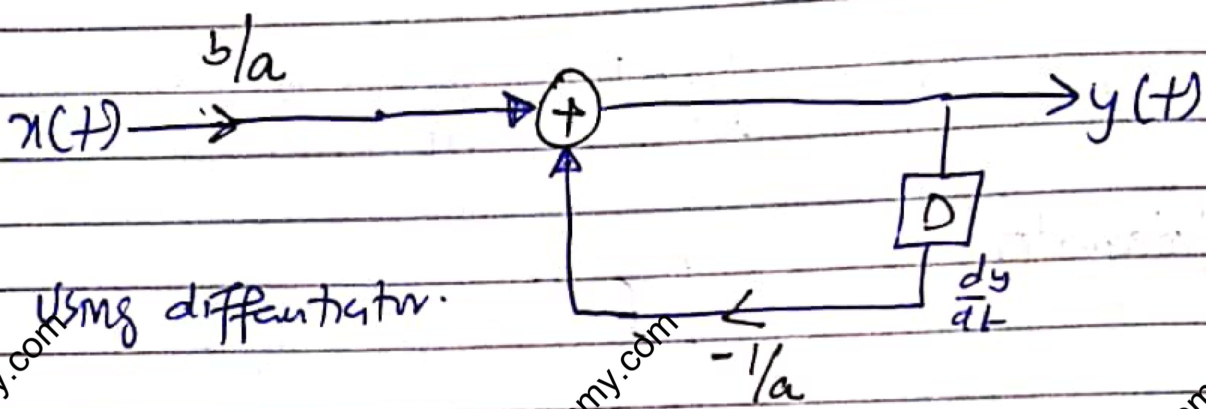
Rearrange such that y[n] is one side and others in the other side.

$$y[n] = b\,x[n] - a\,y[n-1]$$



**Ex.** $\frac{d}{dt} y(t) + a y(t) = b x(t)$

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a} x(t)$$

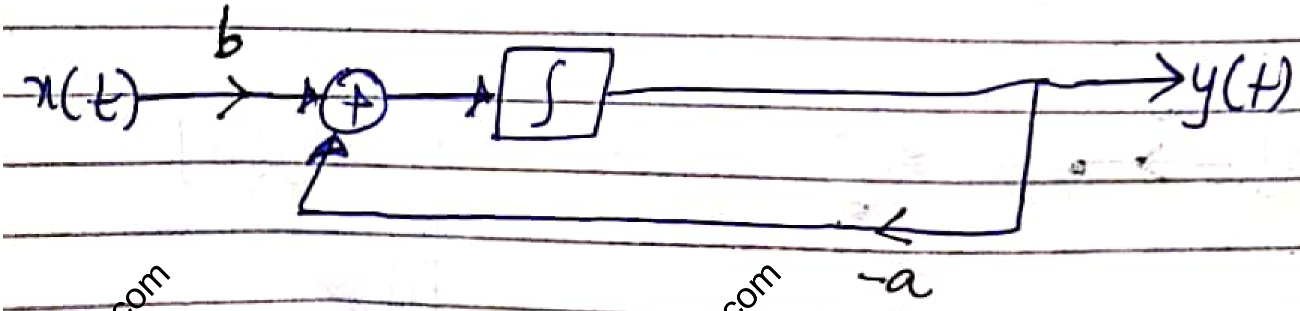



Using differentiator.

Using integrator, we need to rearrange in such a way that the highest order derivative is on one side and all other terms on the other side

$$\frac{d}{dt} y(t) = b x(t) - a y(t)$$

$$y(t) = \int_{-\infty}^{t} \left[ b x(\tau) - a y(\tau) \right] d\tau$$

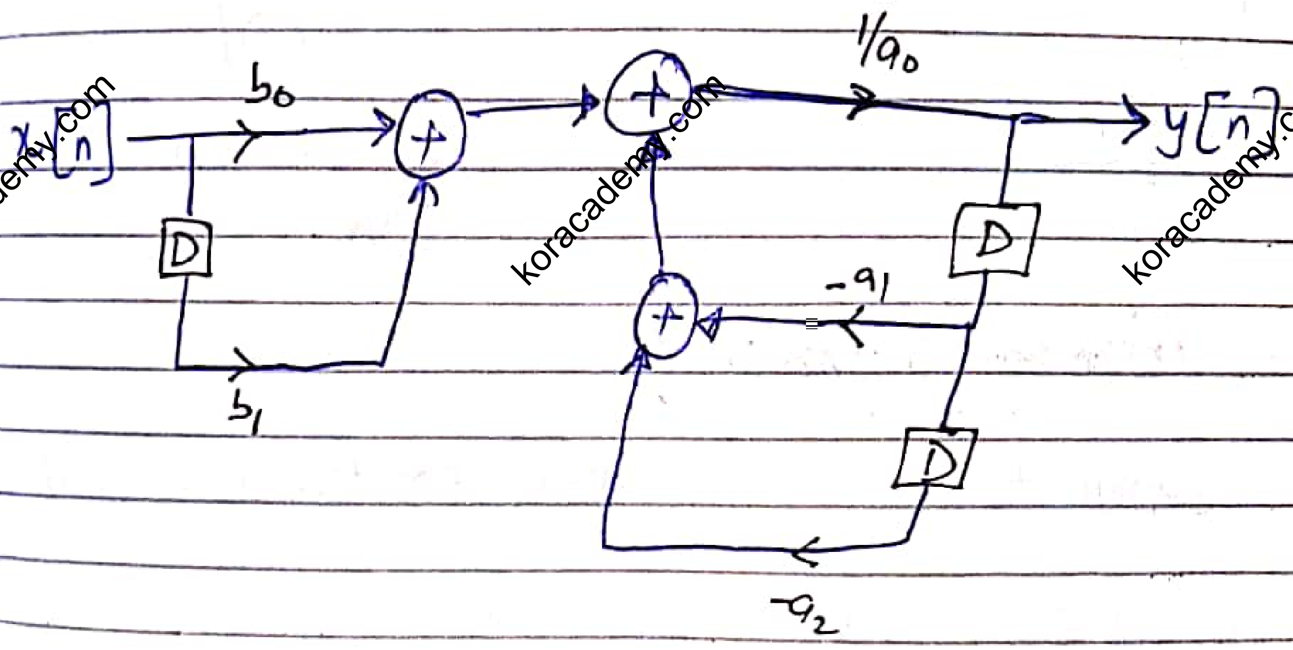


Consider a higher order system;

$$a_0 y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] \cancel{+ b_2 x[n-2]}$$

Rearrange

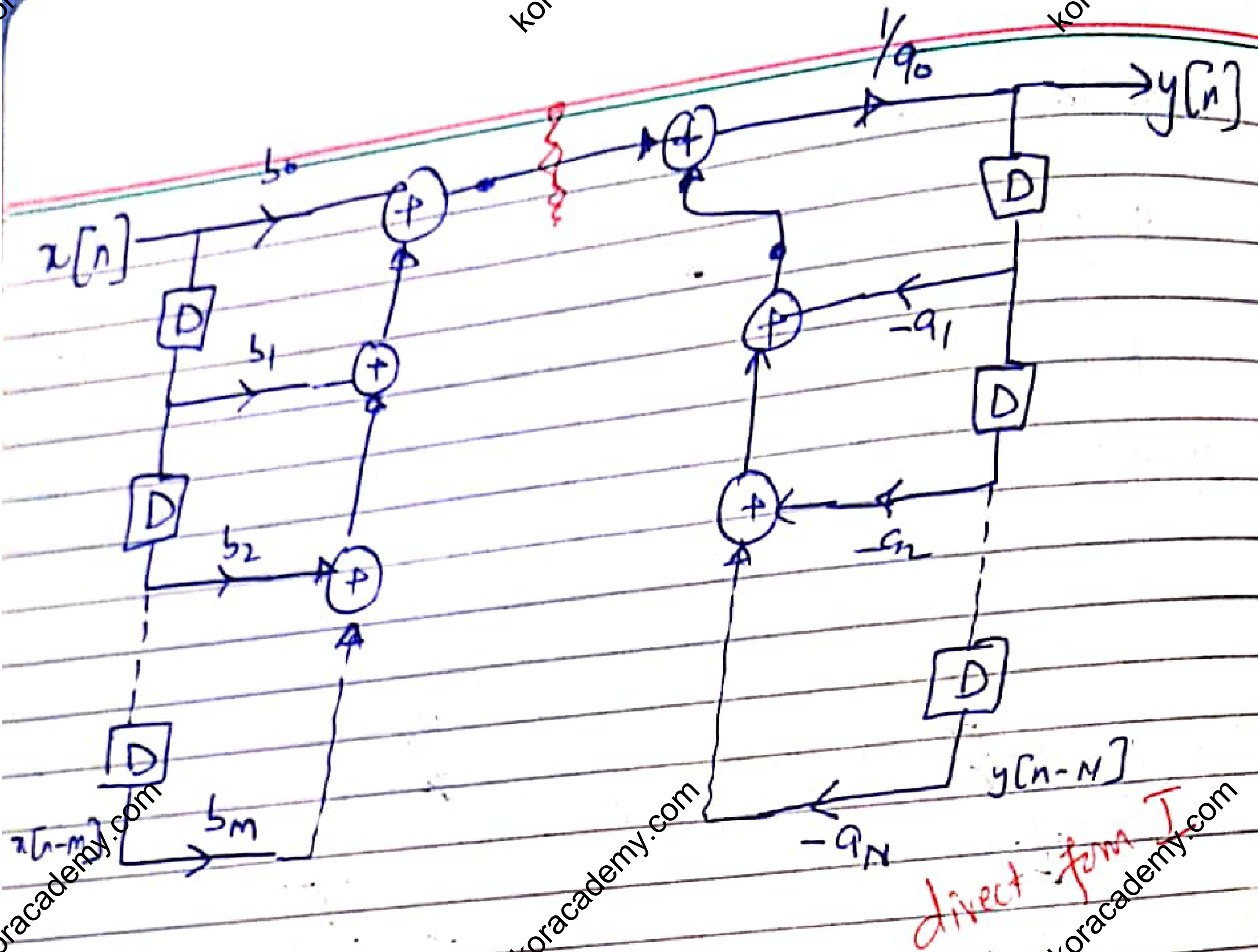
$$y[n] = \frac{1}{a_0}\left\{ b_0 x[n] + b_1 x[n-1] + \cancel{b_2 x[n-2]} - a_1 y[n-1] - a_2 y[n-2] \right\}$$




## $N^{th}$ order systems.

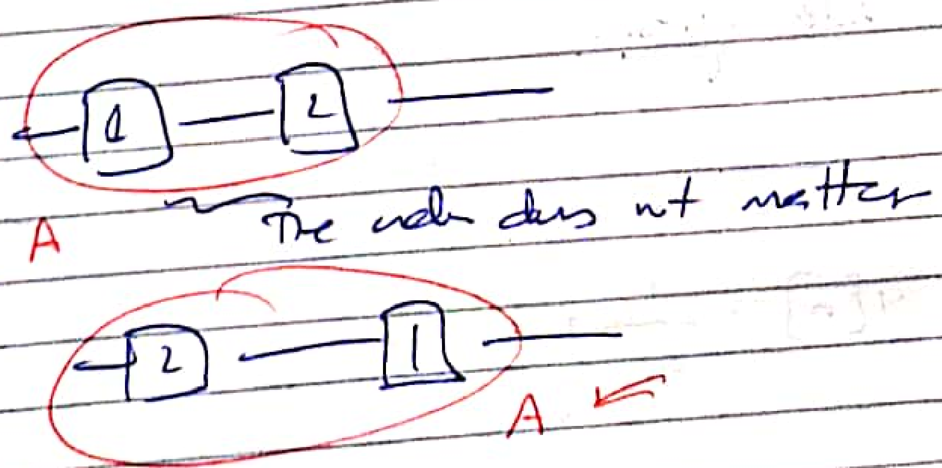
$$\sum_{K=0}^{N} a_K y[n-K] = \sum_{K=0}^{M} b_K x[n-K]$$

$$\Rightarrow y[n] = \frac{1}{a_0}\left\{ \sum_{K=0}^{M} b_K x[n-K] - \sum_{K=1}^{N} a_K y[n-K] \right\}$$

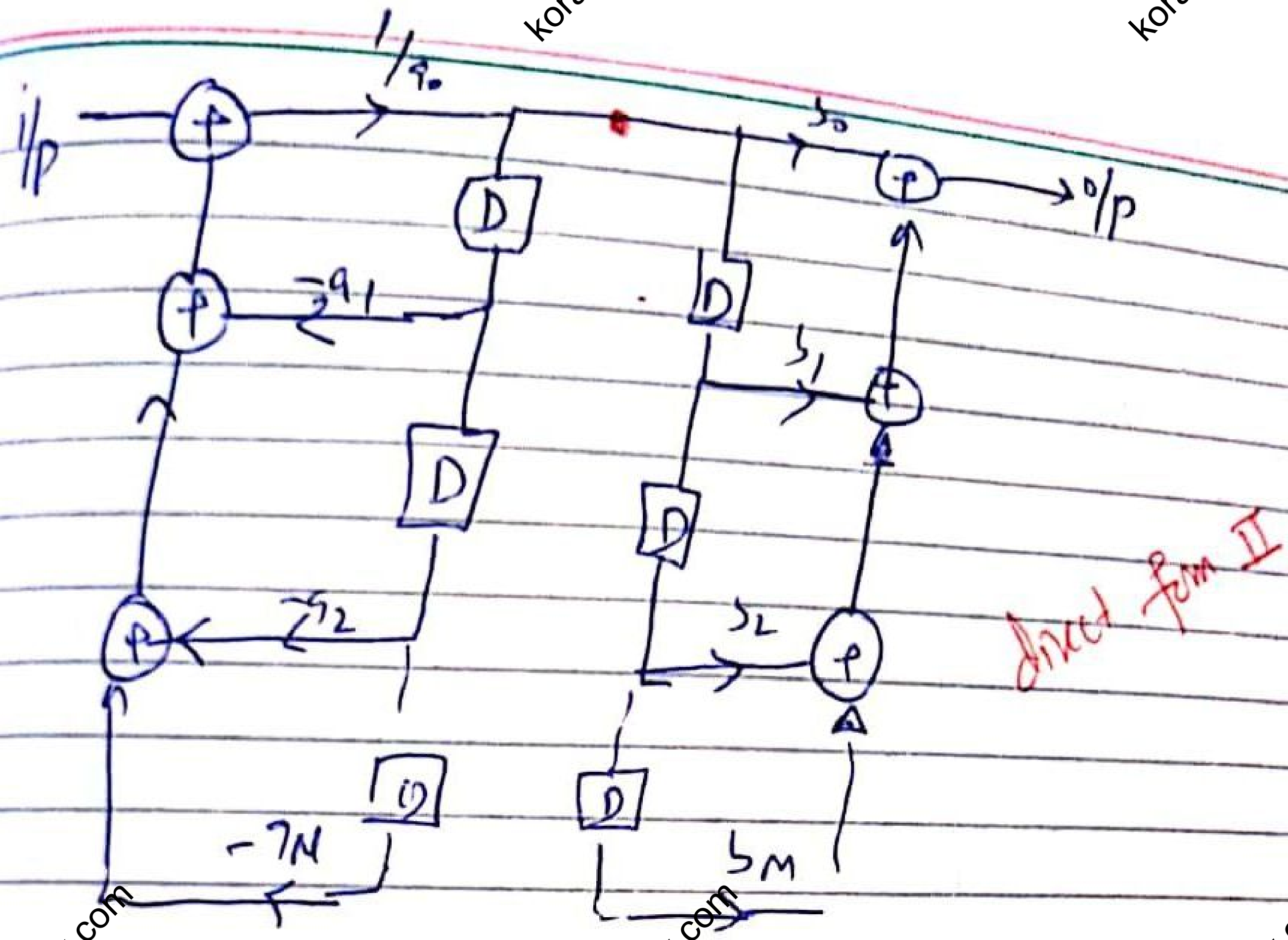


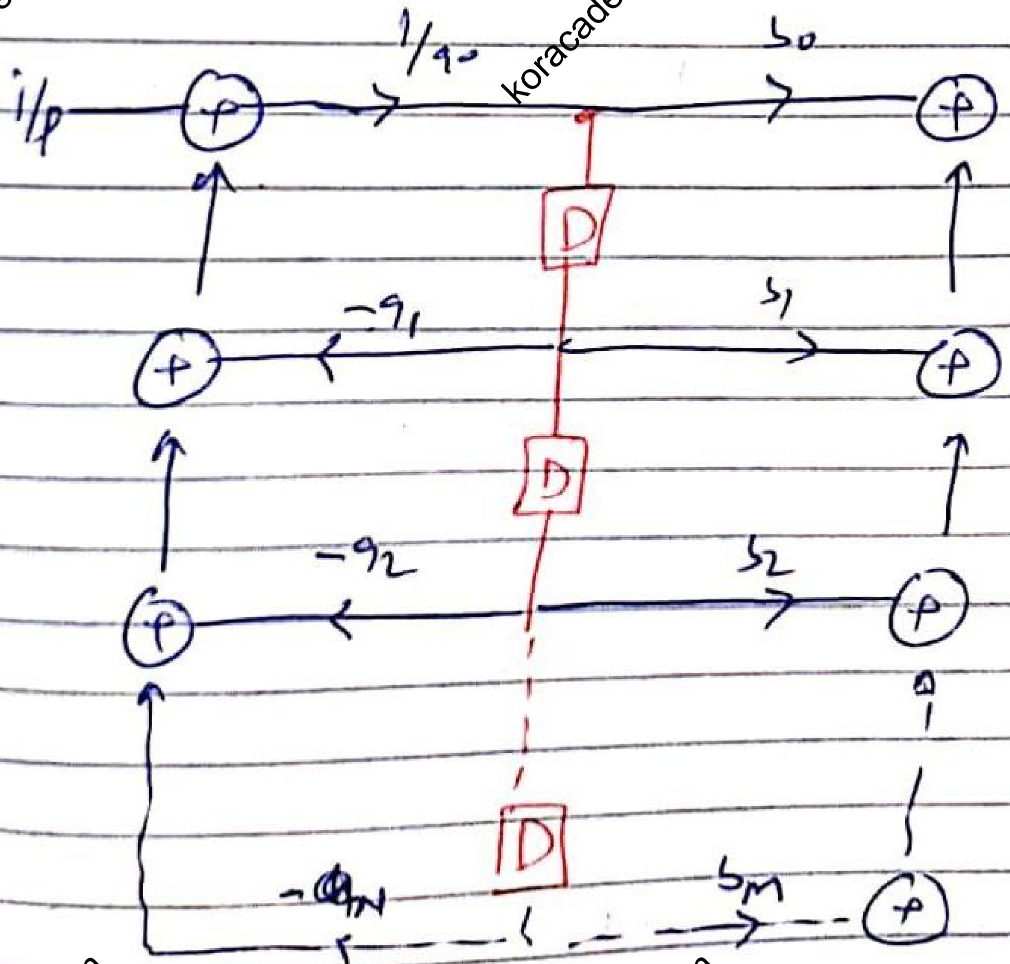

Advantage / Information ?

- combination of two systems connected at a point
  cascade connection.
- Eq B of LTI sys ⇒ Sys 1 = LTI Sys 2 = LTI




So interchanging order of any system;


i/p
1/a0
b0
o/p
D
-a1
b1
-a2
b2
-aN
bM
Direct form II


i/p
1/a0
b0
D
-a1
b1
-a2
b2
-aN
bM

Study CT $s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$

# Chapter 3.

$h(t) = \frac{d\,s(t)}{dt}$

## Fourier Series Representation of Periodic Signals.

Previously our goal was to represent any given signal as a linear combination of some basic signals.

↳ unit impulse → to make analysis simpler

**DT**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

CT $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

## Desirable Features of basic signals.

→ with which we construct more signals.

1. The set of basic signals should be able to construct a large and useful class of signals.
2. The response of an LTI system to each signal should be simple enough to provide us with a convenient representation for the response of the system to any signal which is constructed as a linear combination of the basic signals.

matter for LTI system ... 

$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\underline{\delta(t-\tau)}\,d\tau \rightarrow \boxed{LTI} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,\underline{h(t-\tau)}\,d\tau$

Convolution rep

## Alternate Basic signal.

→ alternate rep → f S rep

### Complex Exponential signal.

CT $\boxed{e^{st}}$ $s$ is complex number. $s = \sigma + j\omega$

DT $\boxed{z^n}$ $z$ is complex

$z = re^{j\omega}$ normally in discrete → polar

$e^{st} \rightarrow \boxed{LTI} \rightarrow H(s)e^{st}$

$z^n \rightarrow \boxed{LTI} \rightarrow H(z)z^n$

A signal for which system o/p is constant (may be complex) times the same signal is known as eigen function of the system.

The constant factor (amplitude scaling factor) is known as eigen value of the system.

As here $e^{st}$, $z^n$ are eigen functions and $H(s)$ and $H(z)$ are eigen values.

Proof:

$$x(t) = e^{st} \rightarrow \boxed{\begin{matrix} LTI \\ h(t) \end{matrix}} \rightarrow y(t) = ?$$

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{st} e^{-s\tau}\, d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

I/P in O/P

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$\hookrightarrow$ depends on impulse response of the system.

$$x(t) = e^{st} \rightarrow y(t) = e^{st} H(s)$$

Similarly in DT;

$$x[n] = \underset{z}{z^n} \rightarrow \boxed{\text{LTI} \; h[n]} \rightarrow y[n] = ?$$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} = \sum_{-\infty}^{\infty} h[k]\, z^{n} z^{-k}$$

$$y[n] = \underset{z}{z^n} \sum_{k=-\infty}^{\infty} h[n]\, z^{-k}$$

I/p in o/p

Eigen value, $H(z) = \sum_{k=-\infty}^{\infty} h[n]\, z^{-k}$

Example for single system $H(z)$ depends on impulse response of the system

eigen function $z^n$.

$$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t)$$

let $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$

lin. comb of exp

For $a_1 e^{s_1 t} \rightarrow a_1 e^{s_1 t} H(s_1)$

$a_2 e^{s_2 t} \rightarrow a_2 e^{s_2 t} H(s_2)$

$a_3 e^{s_3 t} \rightarrow a_3 e^{s_3 t} H(s_3)$

$y(t) = a_1 e^{s_1 t} H(s_1) + a_2 e^{s_2 t} H(s_2) + a_3 e^{s_3 t} H(s_3)$

CT: Input to LTI $x(t) = \sum_k a_k e^{s_k t}$

$$y(t) = \sum_k a_k e^{s_k t} H(s_k)$$

Similarly DT: $x[n] = \sum_k a_k z_k^n$

$$y[n] = \sum_k a_k z_k^n H(z_k)$$

<u>Fourier Series Representation.</u>

It is for periodic signals only.

$e^{st}$, $s = \sigma + j\omega$

In fourier series, we consider $s$ to be purely imaginary.

i.e. $\sigma = 0$ $\quad s = j\omega$

$\Rightarrow e^{j\omega t} \rightarrow$ sinusoidal $\rightarrow$ periodic

$e^{j\omega t} \longrightarrow \boxed{LTI} \rightarrow H(j\omega) e^{j\omega t}$

A linear combination of periodic signals is also periodic

5) CT periodic → $x(t) = x(t+T)$ for all T.

smallest, positive, non zero value of $T$ → fundamental period.

$\omega_0 = \frac{2\pi}{T}$ fundamental frequency

We want to represent this $x(t)$ as a linear combination of $e^{j\omega t}$.

$x(t) = e^{jk\omega_0 t}$ or $x(t) = \cos\omega_0 t$ satisfy properties → periodic

Harmonically related complex exponential.

↳ a number of periodic signals with different fundamental periods but have a period common ↓ fundamental for one

$$\boxed{\phi_k = e^{jk\omega_0 t}}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$k=0 \quad \phi_0(t) = 1$

$k=1 \quad \phi_1(t) = e^{j\omega_0 t} \rightarrow \omega_0 = 2\pi/T \rightarrow T$

$k=2 \quad \phi_2(t) = e^{j2\omega_0 t} \rightarrow 2\omega_0 = \frac{2\pi}{T} \rightarrow T/2$

So the common period $= T$, frequency $= \omega_0$

Take a linear combination of $\phi_k(t)$.

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

$a_k$ ~ quantities

periodic with period $= T$ → generally complex

complex fourier series representation

Exp

An alternate representation → Sin Cos method.

$$\boxed{\text{If } x(t) \text{ is real; } \quad x^*(t) = x(t)}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Take conjugate on both sides.

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-j\omega_0 k t}$$

If $x(t)$ is real;

$$x(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-j\omega_0 k t}$$

Replace $k$ by $-k$ $(-\infty \text{ to } \infty)$

$$x(t) = \sum_{-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

Condition used to be true (P)

for real

$$\boxed{a_k = a_{-k}^*}$$

or $a_k^* = a_{-k}$

$a_k$ → fourier series coefficients OR expansion coeffnt
spectral coeffnts OR

Trigonometric fourier series

Alternate form of fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right]$$

for real $a_k = a_k^*$

$$\Rightarrow x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^* e^{-j\omega_0 k t}\right]$$

conjugate $\quad (a_k e^{j\omega_0 t})^*$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\left[a_k e^{jk\omega_0 t}\right]$$

$a_k$ is generally complex

$a_k = A_k e^{j\theta_k}$ or $a_k = B_k + jC_k$.

consider

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\left[A_k e^{j(\omega_0 t + \theta_k)}\right]$$

$$\boxed{x(t) = a_0 + \sum_{k=1}^{\infty} 2A_k \cos(\omega_0 t + \theta_k)}$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\left[(B_k + jC_k)(\cos k\omega_0 t + j\sin k\omega_0 t)\right]$$

$$\boxed{x(t) = a_0 + \sum_{k=1}^{\infty} 2\left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right]}$$

We mostly use the complex fourier series;

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow ①$$

↳ synthesis equation

$a_k = ?$

Multiply both sides of ① by $e^{-jn\omega_0 t}$

$$x(t) e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 (k-n)t}$$

Take integral on both sides over one period;

$$\int_0^T x(t) e^{-jn\omega_0 t} = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 (k-n)t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j\omega_0 (k-n)t} dt$$

$$\int_0^T e^{j\omega_0 (k-n)t} dt = \begin{cases} T, & k=n \\ 0, & k \neq n \end{cases}$$

periodic → sinusoidal → 0 over one period

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = a_n T$$

→ we find

analysis equation

$$\boxed{a_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt}$$

$n \rightarrow k$

time offset information
→ spectrum, frequency domain

The variables n and k do not matter.

Example

$$x(t) = 1 + \sin w_0 t + 2\cos w_0 t$$

Using Euler's formula

Find simple fs rep $x(t) = \sum a_k e^{jkw_0 t}$

$$x(t) = 1 + \left(\frac{e^{jw_0 t} - e^{-jw_0 t}}{2j}\right) + 2\left(\frac{e^{jw_0 t} + e^{-jw_0 t}}{2}\right)$$

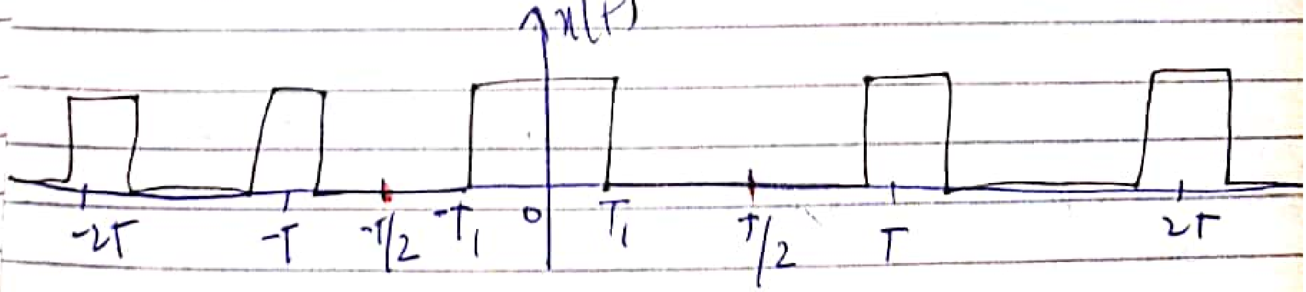
$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{jw_0 t} + \left(1 - \frac{1}{2j}\right)e^{-jw_0 t}$$

So $a_0 = 1$

$$a_{-1} = 1 - \frac{1}{2j} \qquad a_1 = 1 + \frac{1}{2j}$$

$$a_k = 0 \text{ for } |k| > 1$$

Example Periodic square wave




As $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$

we need to find Fourier coefficients ($a_k$) first.

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

If a signal is symmetric, we can also integrate from $-T/2$ to $T/2$ instead of $0$ to $T$.

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1 \cdot 2}{k\omega_0 t}\left(\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right)$$

$$\boxed{a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}}, k \neq 0$$

Originally, $a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt = \frac{1}{T}\int_{-T_1}^{T_1} 1\, dt = \frac{2T_1}{T}$$

$a_0 =$ dc component

As $\omega_0 T = 2\pi$

$$\Rightarrow a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \quad k \neq 0$$

$T_1 = \ldots$

50% duty cycle

If $T = 4T_1 \Rightarrow T_1 = \frac{T}{4}$

$$a_k = \frac{\sin(k\omega_0 T/4)}{k\pi} = \frac{\sin\left(\frac{2\pi}{T}\cdot\frac{T}{4}\right)}{k\pi}$$

$$\boxed{a_k = \frac{\sin(k\pi/2)}{k\pi}}$$

Also $a_0 = \frac{1}{2}$, $a_1 = \frac{1}{\pi}$, $a_{-1} = \frac{1}{\pi}$

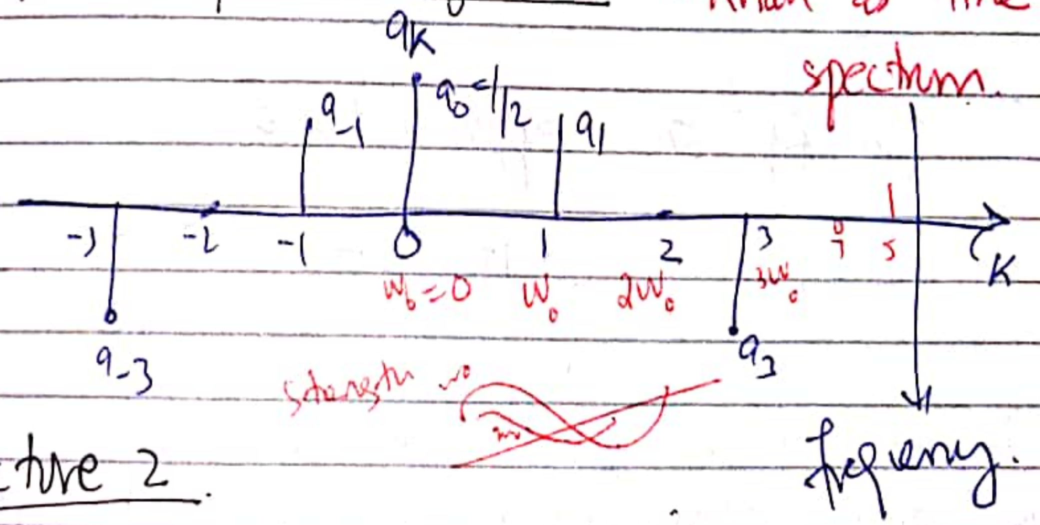
$a_2 = 0$, $a_{-2} = 0$, $a_3 = \frac{-1}{3\pi}$, $a_{-3} = -\frac{1}{3\pi}$

$a_4 = 0$, $a_{-4} = 0$

$a_5 = \frac{1}{5\pi}$, $a_{-5} = \frac{1}{5\pi}$

vertical straight lines

Graphical representation of $a_k$ → Known as line spectrum.




Lecture 2

Continuous time Fourier Series → only for periodic signals.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow \text{Synthesis eq}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\, dt \rightarrow \text{Analysis eq}$$

E Lec 2

Is it possible to represent all periodic signals in terms of fourier series?

Convergence of fourier series.

Say we integrate in a finite interval,

lesser the error, better the approximation.

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

→ truncated or finite fourier series

$$e(t) = x(t) - x_N(t)$$

approximation error (Error signal)

$$e(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

To quantify (size) the approximation error;

$$E_N = \int_T |e(t)|^2 dt$$ → energy over one period

value of e(t) → dependent on $a_k$.

It has been found that in order to minimize the approximation error, the optimized/best value of $a_k$ is;

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

find FS → eqn is still used to find coefficients

The fourier coefficients for finite fourier series remain the same as that of infinite fourier series

as 100 th FS → 91-
200 th FS → [illegible]

If $x(t)$ has fourier series representation, then as N increases, $E_N$ decreases.

approximation improves $\boxed{N \to \infty \quad E_N \to 0}$ → $x_N(t) = x(t)$
→ FS will converge to the original FS.

Two sets of conditions for convergence

(A) over one period, the signal has finite energy then it can be represented by F.S

ie $\int_T |x(t)|^2 dt < \infty$

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

$$e(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

$$\boxed{\int_T |e(t)|^2 dt = 0} \quad \{N \to \infty\}$$

→ This doesn't mean that $x(t)$ and its f.s representation are equal at every value of time.

→ It only means there is no energy in their difference. The avg [illegible] period [illegible] the [illegible]

(B) Dirchlet's conditions

(1) Over one period, $x(t)$ must be absolutely integrable

$$\boxed{\int_T |x(t)| dt < \infty \Rightarrow |a_k| < \infty}$$

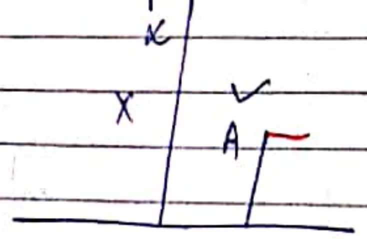
FC should be finite → $x(t)$  → coeff.

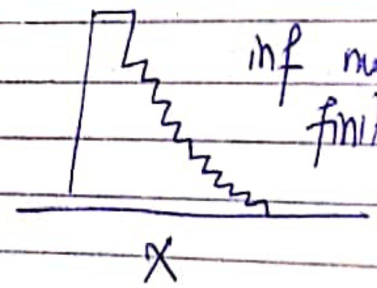
② Over one period, x(t) must have bounded variation i.e finite number of maxima and minima during one period.

③ Over one period, there's are a finite number of discontinuities. Also each of these discontinuities is finite

All useful practical, periodic signals have F.S representation.

→ for example $\sin\left(\frac{2\pi}{t}\right)$ with period 1s, have infinite number of maximas and minimas over a period.

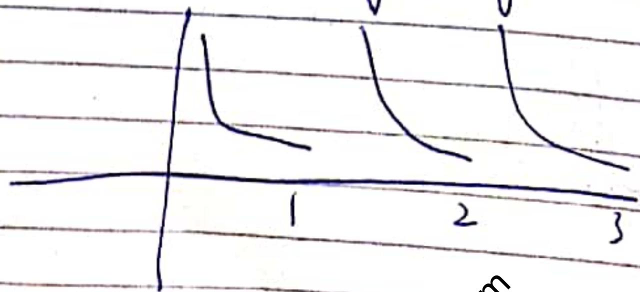



inf number of finite disc.

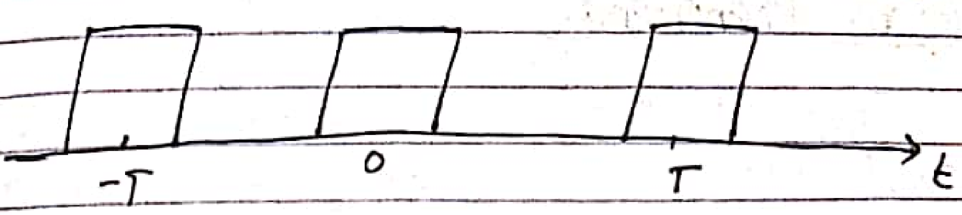
① $x(t) = \frac{1}{t} \quad 0 < t < 1$ ? → $T=1$

is not absolutely integrable → no fourier series.



Sinusoids → derivation ?

Consider the periodic square wave; → discontinuous.




Fourier believed that fourier series existed for this signal.

Michelson constructed Harmonic analyzer.
experiments

i/p (periodic signals) → Harmonic analyzer → o/p

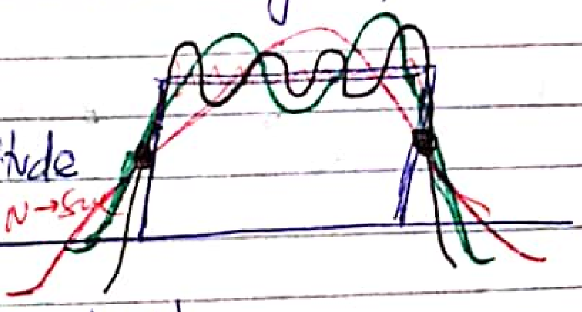
function = Finite F.S representation.

1898 number of terms ← N → 80 (options)

For continuous signals, results were acurate and according to Fourier.

For discontinuous signals, results were strange.

Points of concern
→ ripples
→ ripple amplitude
↳ whether N→ ...




→ Behaviour at point of discontinuity.

$\sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$

Michelson was unable to explain.

So he presented it to Josiah Gibbs.

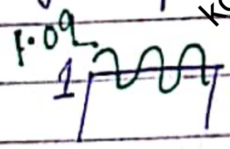
## Gibb's Phenomena

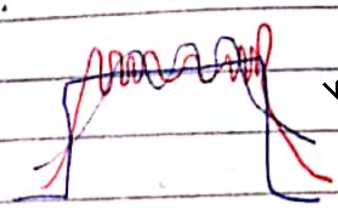
① Truncated F.S approximation of a discontinuous signal x(t) will in general exhibit high frequency ripples and overshoot near its discontinuity.

② The peak values of these ripples does not increase with increasing N.

③ The peak vale / overshoot is always 9% of the height of discontinuity.

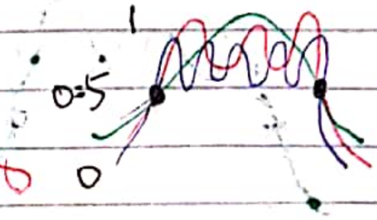
eg



④ As N increases, the ripples will compress towards point of discontinuity.

⑤ At point of discontinuity, the finite F.S converges to the average value of the original signal x(t) on either side of the discontinuity.



# Properties of CT F.S

Notation,

$$x(t) \xleftrightarrow{F.S} a_k$$

$x(t)$ is paired with its fourier coefficients.

## ① Linearity

$$x(t) \xleftrightarrow{F.S} a_k$$
$$y(t) \xleftrightarrow{F.S} b_k$$

$$\boxed{z(t) = A\,x(t) + B\,y(t) \xleftrightarrow{F.S} C_k = A a_k + B b_k}$$

If a signal is linear combination of two signals, its fourier coefficients will also be linear combination of their corresponding coefficients.

## ② Time Shifting.

$$x(t) \xleftrightarrow{F.S} a_k$$

$$\boxed{x(t-t_0) \xleftrightarrow{F.S} b_k = e^{-jk\omega_0 t_0} a_k}$$

As $$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

let $$b_k = \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

let $\tau = t - t_0 \Rightarrow t = \tau + t_0$

$$\Rightarrow b_k = \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0 \tau}\, e^{-jk\omega_0 t_0}\, d\tau$$

$$b_k = e^{-jk\omega_0 t_0} \cdot \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau$$

$a_k$

$$\Rightarrow b_k = e^{-jk\omega_0 t_0}\, a_k$$

③ Time Reversal

$$x(t) \xleftrightarrow{FS} a_k$$

$$\boxed{x(-t) \xleftrightarrow{FS} a_{-k}}$$

The synthesis equation;

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t}$$

Replace k by -k;

$$x(-t) = \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\omega_0 t}$$

Mirror image of func. coffients.

→ If $x(t)$ is an even signal;

ie $x(t) = x(-t)$

$$\Rightarrow a_K = a_{-K} = \text{even}$$

→ If $x(t)$ is an odd signal;

ie. $x(t) = -x(-t)$

$$\Rightarrow a_K = -a_{-K} = \text{odd}$$

④ Time Scaling

$T, \omega_0 \leftarrow x(t) \xleftrightarrow{F.S} a_K$

$T' = \frac{T}{\alpha}$, $\omega = \frac{2\pi\alpha}{T}$

$T\downarrow f\uparrow = |\alpha|>1$ compress

$|\alpha|<1$ expand

$\alpha\omega_0, \frac{T}{\alpha} \leftarrow x(\alpha t) \xleftrightarrow{F.S} a_K$

As
$$x(t) = \sum_{K=-\infty}^{\infty} \underline{a_K}\, e^{jK\omega_0 t}$$

$$x(\alpha t) = \sum_{K=-\infty}^{\infty} \underline{a_K}\, e^{jK\omega_0 \alpha t}$$

→ The fourier coefficients will remain the same but the fourier series representation will change.

The position of the coefficient in the spectrum will change. $(\alpha\omega_0)$

⑤ Multiplication

$x(t) \xleftrightarrow{F.S} a_K$

$y(t) \xleftrightarrow{F.S} b_K$

$$\boxed{z(t) = x(t)y(t) \xleftrightarrow{F.S} C_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}}$$

convolution

⑥ Conjugate and conjugate symmetry

$$x(t) \xleftrightarrow{F.S} a_k$$

$$\boxed{x^*(t) \xleftrightarrow{F.S} a^*_{-k}}$$

As $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

Replace k by -k,

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

If $x(t)$ is real;

$$x(t) = x^*(t)$$

$\Rightarrow a_k^* = a_{-k}$ → conjugate symmetry

or $a_{-k}^* = a_k$

eg → $a_1 = 1 + j5$

$a_1^* = 1 - j5$

→ $a_{-1}$

$a_k^* = a_{-k}$ $\quad a_k \rightarrow$ complex

Can be written as;

$|a_k| \measuredangle a_k$ or $Re\{a_k\} + j\,Im\{a_k\}$

$\rightarrow |a_k| = |a_{-k}|$ or $Re\{a_k\} = Re\{a_{-k}\}$

$\measuredangle a_k = -\measuredangle a_k$ $\quad Im\{a_k\} = -Im\{a_{-k}\}$

Graph of $a_k$;

mag $|a_k|$ $\rightarrow k$

Phase $\measuredangle a_k$ $\rightarrow k$

$Re\{a_k\}$ $\rightarrow k$

$Im\{a_k\}$ $\rightarrow k$

⑦ Differentiation

$$x(t) \xleftrightarrow{F.S} a_k$$

$$\boxed{y(t) = \frac{d}{dt} x(t) \xleftrightarrow{F.S} b_k = jk\omega_0 a_k}$$

⑧ Integration

$$\boxed{y(t) = \int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{F.S} b_k = \frac{1}{jk\omega_0} a_k}$$

# Parseval's Relation

Say $x(t)$ with period $T$.

$$\text{Avg power} = \frac{1}{T}\int_T |x(t)|^2\, dt$$

So according to Parseval;

$x(t)$ if periodic $\rightarrow$ has F.S $\rightarrow$ has F. coefficient.

$$x(t) \xleftrightarrow{F.S} a_K$$

$$x(t) = \sum_{k=-\infty}^{\infty} |a_k| e^{-jk\omega_0 t}$$

$$\boxed{\text{Avg power} = \sum_{K=-\infty}^{\infty} |a_K|^2}$$

## Discrete Time Fourier Series

Periodic signals in discrete time,

$$x[n] = x[n+N], \text{ for all } N$$

$N \rightarrow$ fundamental period

$\frac{2\pi}{N} = \omega_0 \rightarrow$ fundamental frequency

Representation of this periodic signal with period $N$ in terms of complex exponential signals (periodic).

$$e^{j\omega_0 n} = e^{j\frac{2\pi}{N}n} \rightarrow \text{periodic exponential signal with period } N.$$

$\phi_k(t) = e^{jk\omega_0 t}$ → for every different value of k, we get a different periodic signal.

$\phi_k[n] = e^{jk\omega_0 n}$ → change value of k. → As N number of values of k are completed, the signal will start repeating itself.

## Harmonically related exponential signals;

$$\phi_k[n] = e^{jk\omega_0 n}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\phi_0[n], \phi_1[n], \phi_2[n], \phi_3[n] \ldots \phi_{N-1}[n], \phi_N[n] + \phi_{N+1} + \phi_{N+2} + \ldots$$

These two will be the same.
So repetition starts.
N number of unique signals.
not necessary that 1 start from k=0 — If start from k=0 → N−1; If 1 → N

Linear combination of harmonically relation exp signals;

$$x[n] = \sum_{k} a_k \phi_k[n].$$

↳ All of these are periodic with period N.

$$\phi_k[n] = e^{jk\omega_0 n}$$

$$\boxed{\phi_k[n] = \phi_{k+rN}[n]}$$

r → multiple.

ie $\phi_0 = \phi_N = \phi_{2N} = \phi_{3N}$

Linear combination;

$$\boxed{x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}}$$

<> means the start and end doesn't matter.
↳ consider N number of values.

0 — N−1
1 — N
2 — N+1

→ D.T F.S → Synthesis equation

In CT FS the summation is over an infinite range. (infinite no. of exp terms)

In DT → finite interval of summation.
↳ there are no convergence issues as in CT FS.

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n}$$

the series will always converge to the original signal.

Coefficient, $a_k = ?$

Using the same procedure as in CT,

$$\boxed{a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n}}$$

Analysis equation

In CT FS,

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t}$$

The difference is all s/c of different nature of harmonically related exponentials.

Consider; $x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n]$.

let $x[n] = \sum_{k=0}^{N-1} a_k \phi_k[n]$.

Expanding,

$$x[n] = \boxed{a_0 \phi_0[n]} + a_1 \phi_1[n] + a_2 \phi_2[n] + \cdots + a_{N-1} \phi_{N-1}[n]$$

or say $x[n] = \sum_{k=1}^{N} a_k \phi_k[n]$.

Expanding

$$x[n] = a_1 \phi_1[n] + a_2 \phi_2[n] + \cdots + \boxed{a_N \phi_N[n]}$$

Different terms, but we know that; $\phi_0 = \phi_N$

$\Rightarrow a_0 = a_N$

Generalizing

$$\boxed{a_k = a_{k+N}}$$

$a_0 = a_N$

$a_1 = a_{N+1}$

→ Fourier coefficients of DT FS repeat themselves after every N interval.

↳ this behaviour was not shown by CT FS.

↳ ie $a_k$ are periodic with period $N$.

# Lecture 3

## DT Fourier Series

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \rightarrow \text{Synthesis}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\frac{2\pi}{N}n} \rightarrow \text{Analysis}$$

In DT, summation is over finite number of terms whereas in CT it over infinite interval.

$a_k = a_{k+N}$

The fourier coefficients in DT are periodic in nature.

**Example** $\quad x[n] = \sin \omega_0 n \quad$ DT FS?

Sin in DT may or may not be periodic.
Checking periodicity;

↳ (i) If $\frac{2\pi}{\omega_0} =$ integer $= N \rightarrow$ period.

Say fulfilling;

$$\Rightarrow \quad x[n] = \sin\left(\frac{2\pi}{N}\right)n$$

We want to represent $x[n]$ in this form;

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$$

Using Euler's relation,

$$x[n] = \frac{1}{2j} e^{j\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j\left(\frac{2\pi}{N}\right)n}$$

What is $\frac{1}{2j}$ and $-\frac{1}{2j}$ representing?

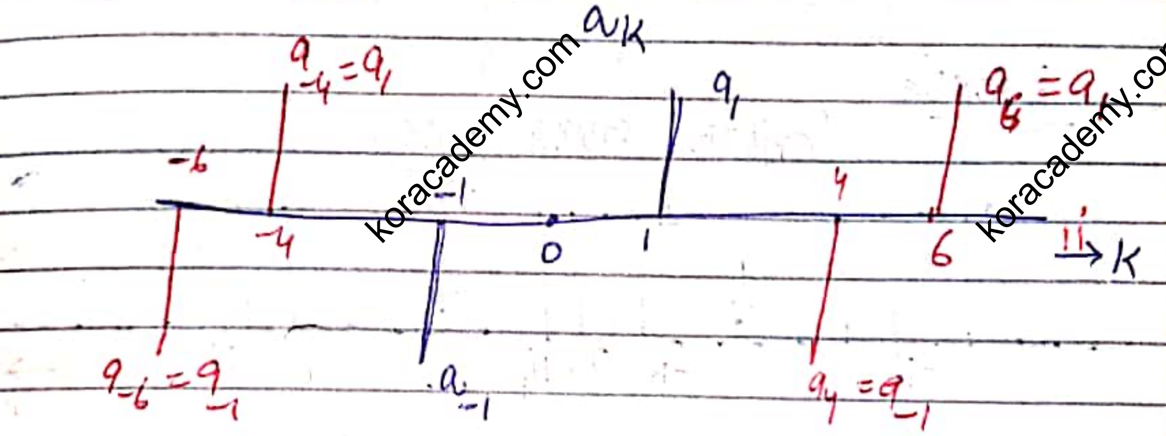
$\downarrow$ $a_1$ $\quad$ $\downarrow$ $a_{-1}$

The rest coefficients are zero.

$$a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j}$$

$$a_k = 0, \quad k \neq \pm 1$$

Assume $N = 5$. periodic




ii. Consider the other possibility for periodicity; period

ie if $\frac{2\pi}{\omega_0} \neq$ ratio of integers $= \frac{N}{M}$

$$\Rightarrow \omega_0 = \left(\frac{2\pi}{N}\right)M$$

$$x[n] = \sin M\left(\frac{2\pi}{N}\right)n$$

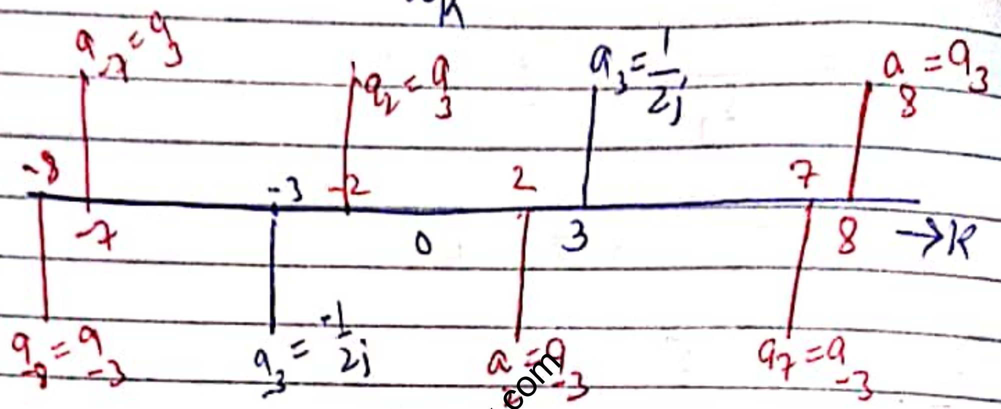
Again using Euler's relation:

$$x[n] = \underbrace{\frac{1}{2j}}_{a_m} e^{+jm\left(\frac{2\pi}{N}\right)n} - \underbrace{\frac{1}{2j}}_{a_{-m}} e^{-jm\left(\frac{2\pi}{N}\right)n}$$

→ periodic

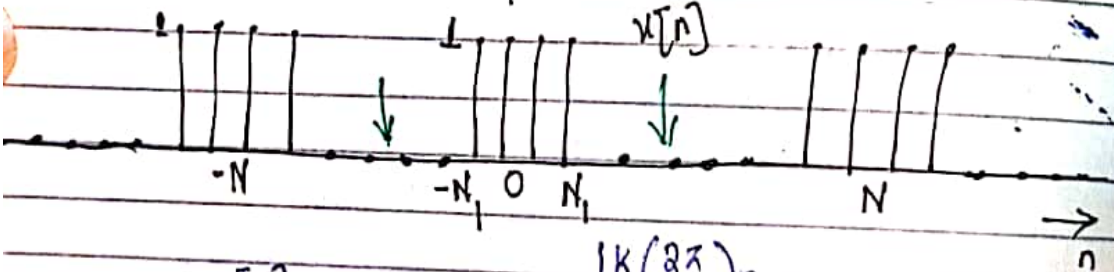
Assume $N=5,\ M=3$

$$\Rightarrow \frac{2\pi}{\omega_0} = \frac{N}{M} = \frac{5}{3}$$




Example

Periodic square wave.




$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} = ?$$

As $$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

but non zero only $-N_1 \le n \le N_1$ ← interval say $-\frac{N}{2}+1$ to $\frac{N}{2}$

So $$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} (1)\, e^{-j\frac{2\pi}{N}nk}$$

modification (make lower limit zero.)

Let $m = n + N_1$

$$\Rightarrow a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\frac{2\pi}{N}(m - N_1)}$$

$$a_k = \frac{1}{N} e^{jk\left(\frac{2\pi}{N}\right)N_1} \sum_{m=0}^{2N_1} \underbrace{\left(e^{-jk\left(\frac{2\pi}{N}\right)}\right)^m}_{x}$$

As $\displaystyle\sum_{m=0}^{N} x^m = \frac{1 - x^{N+1}}{1 - x}$

$$\Rightarrow a_k = \frac{1}{N} e^{jk\left(\frac{2\pi}{N}\right)N_1} \left[ \frac{1 - e^{-jk\left(\frac{2\pi}{N}\right)(2N_1+1)}}{1 - e^{-jk\frac{2\pi}{N}}} \right]$$

After simplification

Repeat 1 in computer

$\hookrightarrow e^{+j\theta} - e^{-j\theta} \rightarrow \sin\theta$

$$a_k = \frac{1}{N} \left[ \frac{\sin\left\{ \frac{2\pi}{N} k \left(N_1 + \frac{1}{2}\right) \right\}}{\sin\left(\frac{\pi k}{N}\right)} \right] \quad \text{For } k \neq 0$$

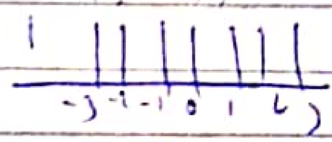
$a_0 = ?$

$\hookrightarrow$ is the DC or average value (no. of samples)

Over one period, how much is the duration for which signal is nonzero.

$$a_0 = \frac{2N_1 + 1}{N}$$

$N_1, -N_1, 0$




$2N_1 + 1 = 6 + 1 = 7$

## Properties of DT. FS

1. **Linearity** similar as CT.

$$x[n] \xleftrightarrow{FS} a_k$$

$$y[n] \xleftrightarrow{FS} b_k$$

$$\boxed{A\,x[n] + B\,y[n] = z[n] \xleftrightarrow{F.S} C_k = Aa_k + Bb_k}$$

Not only for two, for n number of signals.

## 2. Time Reversal

$$x[n] \xleftrightarrow{F.S} a_k$$

$$\boxed{x[-n] \xleftrightarrow{F.S} a_{-k}}$$

As synthesis eq, $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$

So $x[-n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$ → This should be +.

Replace k by -k

$$x[-n] = \sum_{k=\langle N\rangle} a_{-k} e^{jk\omega_0 n}$$

## 3. Time Shifting.

$$x[n] \xleftrightarrow{F.S} a_k$$

$$\boxed{x[n-n_0] \xleftrightarrow{F.S} e^{-jk\omega_0 n_0} a_k}$$

Proof As $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$

Hence $x[n-n_0] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 (n-n_0)}$

## 9. Conjugate and Conjugate Symmetry

$$x[n] \xleftrightarrow{F.S} a_k$$

$$\boxed{x^*[n] \xleftrightarrow{F.S} a^*_{-k}}$$

If $x[n]$ is real; $\quad x[n] = x^*[n]$

$$\Rightarrow a_k = a^*_{-k} \rightarrow \text{conjugate symmetric.}$$

As $a_k$ is complex $\quad Re(a_k) = Re(+a_{-k})$

$$Im(a_k) = -Im(a_{-k})$$

Similarly in polar form

$$|a_k| = |a_{-k}| \quad \text{and} \quad \measuredangle a_k = -\measuredangle a_{-k}$$

If $x[n]$ is real and even;

$$x[n] = x[-n]$$

$$a_k = a_{-k}$$

$$\Rightarrow a^*_k = a_k \qquad \text{as } a_k = a_{-k}$$

$\hookrightarrow a_k$ is real and even.

If $x[n]$ is real and odd;

$$x[n] = -x[-n]$$

$a_k$ is purely imaginary and odd.

5. First difference

Differentiation in CT

$x(t) \longleftrightarrow a_k$

$\frac{d}{dt} x(t) \longleftrightarrow jw_0 k\, a_k$

$$x[n] \xleftrightarrow{FS} a_k$$

$$x[n] - x[n-1] = y[n] \longleftrightarrow b_k = a_k - e^{-jkw_0} a_k$$

$$\boxed{b_k = a_k\left(1 - e^{-jkw_0}\right)}$$

6. Multiplication

$$x[n] \xleftrightarrow{FS} a_k \qquad y[n] \xleftrightarrow{FS} b_k$$

$$\boxed{x[n].y[n] \xleftrightarrow{FS} \sum_{l=\langle N\rangle} a_l\, b_{k-l}}$$

↳ convolution periodic

Parseval's Relation

$x[n]$ periodic with period $N$.

$$\text{Avg power} = \frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2$$

As $x[n] \xleftrightarrow{FS} a_k$

$$\text{Parseval average power} = \sum_{k=\langle N\rangle} |a_k|^2$$

# The Continuous time Fourier Transform.

1. For periodic signals we have FS which represent signals in terms of complex exponential signals ($e^{jk\omega_0 t}$). (linear combination)

2. We can also represent Aperiodic signals in terms of complex exponential signal → Fourier transform.

Q. What is the difference if both periodic and aperiodic signals can be represented in terms of exponentials?

3. For periodic signals the complex exponential signals are harmonically related.

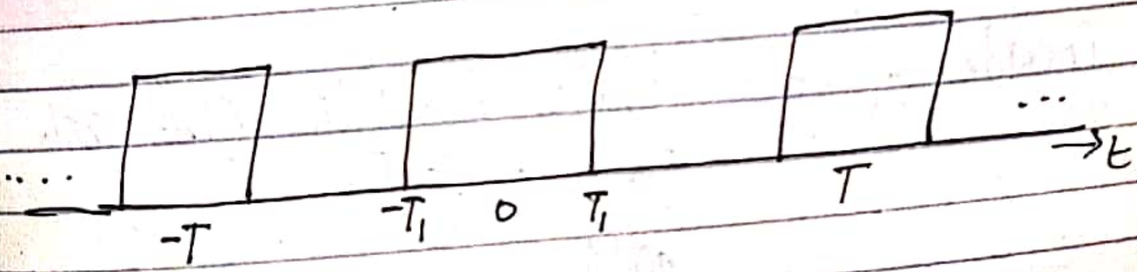
4. For aperiodic signals, the comp exp signals are infinitesimally close in frequency and not harmonically related.

very small

5. Fourier reasoned that any Aperiodic signal can be viewed as periodic signal with infinite period. (consider a periodic signal ⇒ FS → increase the period
FT ∝ ← effect ← period

! Main concept for development

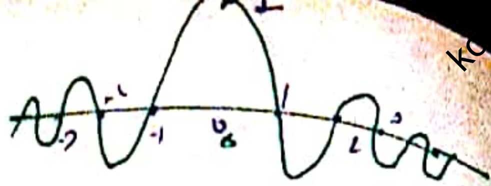
## Development of Fourier transform:



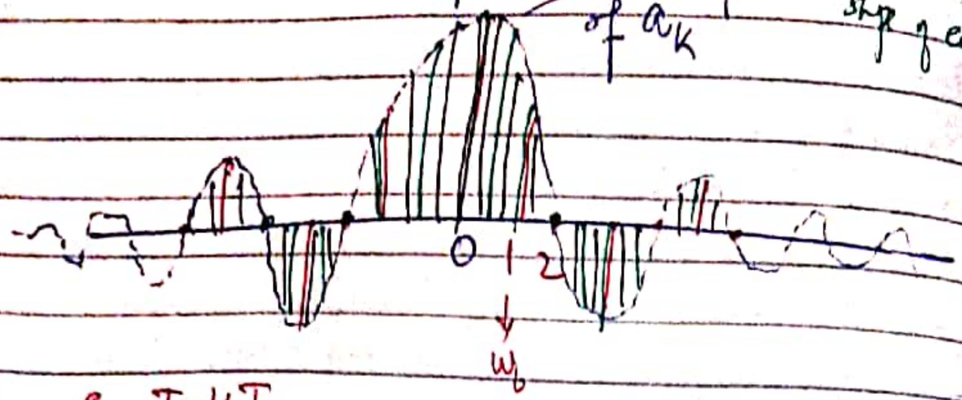

FS coefficients;

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \rightarrow \text{sinc function}$$

$$\text{sinc}\,\theta = \frac{\sin(\pi\theta)}{\pi\theta}$$



Assume $T_1$ is fixed. → envelope will remain constant → no change in shape of envelope




say for $T = 4T_1$

say $T = 8T_1$ — T increase → $\omega_0$ denser.

↳ compress → close to each other

say $T = 16T_1$

↳ more closer → getting congested.

$a_k$ are like samples of envelope

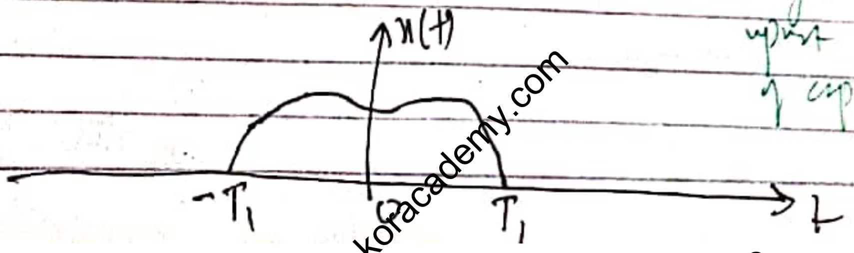
As period ↑ ⇒ sampling ↑

If we approach the period to $\infty$, $a_k$ would approach the envelope.

So the discrete nature of coefficients would convert into a continuous one.

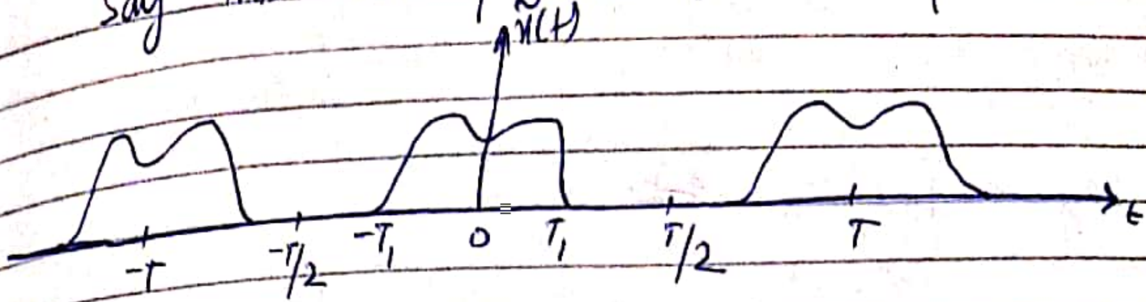
Consider a finite duration Aperiodic signal such that $x(t) = 0$ for $|t| > T_1$.

Say we want to represent this in terms of complex exponential.



Idea of fourier → Aperiodic → some periodic

Say this is a periodic signal; with period T.




$\tilde{x}(t)$ is periodic version of $x(t)$.

$x(t)$ and $\tilde{x}(t)$ are equal in one period.

or $x(t)$ is one period of $\tilde{x}(t)$!

As we increase $T$ of $\tilde{x}(t)$, the time for which $x(t)$ and $\tilde{x}(t)$ are equal will increase.

If we make $T = \infty$ for $\tilde{x}(t)$, the two signals will be exactly equal for all time.

F.S of $\tilde{x}(t)$ ; $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ → (1)

$$a_k = \frac{1}{T} \int_T \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

in this interval $x(t)$ and $\tilde{x}(t)$ are same:

$$\Rightarrow a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-\boxed{jk\omega_0}t}$$

$-\infty \dashrightarrow$ b/c signal is zero otherwise

Say a function; ~~even~~ Envelope function;

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad (\omega = k\omega_0)$$

$$\boxed{a_k = \frac{1}{T} X(jk\omega_0)} \rightarrow \text{in } ①$$

$$\Rightarrow \tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0)\, e^{jk\omega_0 t}$$

As $\omega_0 = \frac{2\pi}{T} \Rightarrow \frac{\omega_0}{2\pi} = \frac{1}{T}$

$$\Rightarrow \boxed{\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0}$$

Increase the period $T$ and see what happens when $T \rightarrow \infty$.

k times small value, the difference will be so small that we will call it a continuous variable

$\tilde{x}(t) = x(t)$, $\omega_0 \rightarrow 0$, $k\omega_0 \rightarrow \omega$

$\sum \rightarrow \int$

Inverse FT $\leftarrow$ $$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega}$$

Synthesis eq

$a_k e^{j\omega_0 t}$ weight $\leftarrow \frac{X(j\omega)\, d\omega}{2\pi}\, e^{j\omega t}$

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$ → Forward FT → Analysis eq.

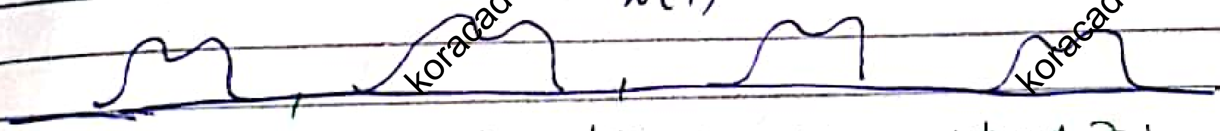
imp point: $a_K = \overline{\frac{1}{T} \oint X(jK\omega_0)}$ = another point of view

or $a_K = \frac{1}{T} X(j\omega)\Big|_{\omega = K\omega_0}$ for different values of $k$

→ It appears as if fourier coefficients are samples of fourier transform ←

→ Say we have a periodic signal; (more form)

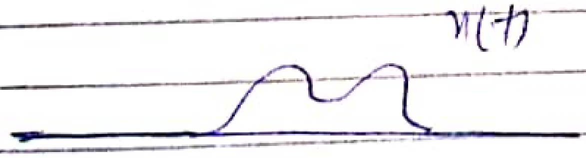
$\tilde{x}(t)$



$a_K = ?$ we can use the analysis eq, but we interpret it by another method.

Say we take one period of the periodic signal;

$x(t)$



↳ so this is Aperiodic. ↵

FS x FT → $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

finding $a_K$ of $\tilde{x}(t)$ with the help of

↳ $a_K = \frac{1}{T} X(j\omega)\Big|_{\omega = K\omega_0}$

~~Solve examples~~

# Convergence of FTCT

Not all Aperiodic signals will have FT.

Assume $X(j\omega)$ exists for some signals, say $x(t)$

IFT $$\tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Is $\tilde{x}(t)$ a valid representation of our original signal?

Error → $e(t) = \tilde{x}(t) - x(t)$

→ (quantization parameter)

We decide on the basis of energy of error signal.

$$\text{Energy} = \int_{-\infty}^{\infty} |e(t)|^2 dt = 0 \quad \text{converge}$$

**Condition For FT to exist**

1- $x(t)$ has finite energy.

ie $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$

→ $\int_{-\infty}^{\infty} |e(t)|^2 dt = 0$

## Dirichlet's Condition for Convergence.

① $x(t)$ must be absolutely integrable.

ie $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

For FS we … while FT we use … period sig → A → period

② $x(t)$ is having finite number of maximas and minimas during any finite time interval.

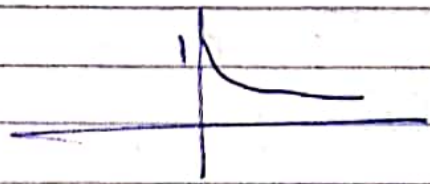
③ $x(t)$ is having finite number of finite discontinuities within any finite interval.

### Example.

$x(t) = e^{-at}.u(t)$ , $a > 0$

↳ exponentially decaying → Aperiodic

FT $X(j\omega) = ?$



As $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$X(j\omega) = \int_0^{\infty} e^{-at} . e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$\left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty}$$

$$= \frac{-1}{a+j\omega}\left[e^{-\infty} - e^{0}\right]$$

$e^{-\infty} \to 0$, $e^{0} \to 1$

$(-)$

$$\boxed{X(j\omega) = \frac{1}{a+j\omega}, \quad a > 0}$$

↳ Generally a complex function.

↳ in polar form.

$|X(j\omega)| \angle X(j\omega)$

mag ← ↳ phase

$= \frac{1}{a+j\omega}$

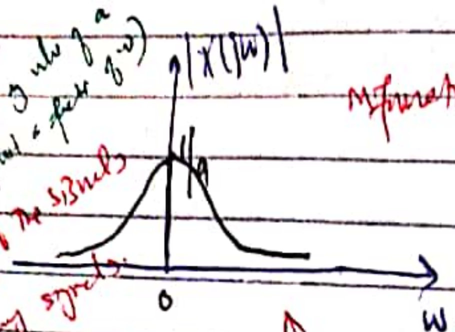
$$|X(j\omega)| = \frac{1 \rightarrow \text{real phase gt } 0}{\sqrt{a^2+\omega^2}}$$

$$\angle X(j\omega) = 0 - \tan^{-1}\left(\frac{\omega}{a}\right)$$

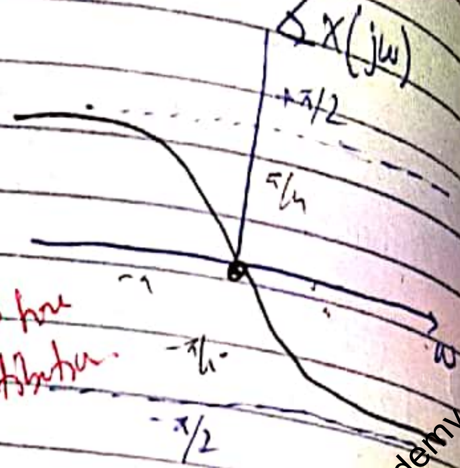
matters. (select a value of a and plot a specific graph)

most of the signals are low frequency signals.

Information?




Mag spectrum. ↑ higher frequencies have less contribution.



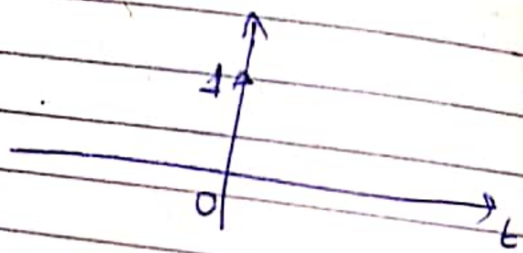

Spectrum → frequency components.

Phase spectrum

Example

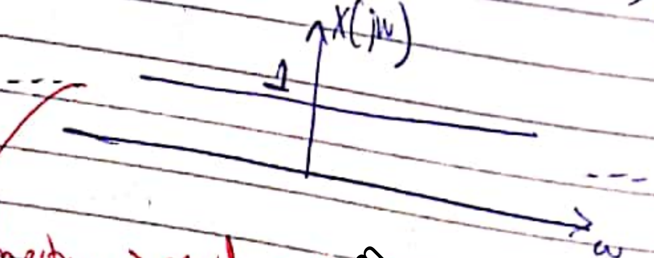
$x(t) = \delta(t)$ → Aperiodic.

CTFT = ?




$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$\Rightarrow X(j\omega) = 1 \rightarrow$ constant $\rightarrow$ real



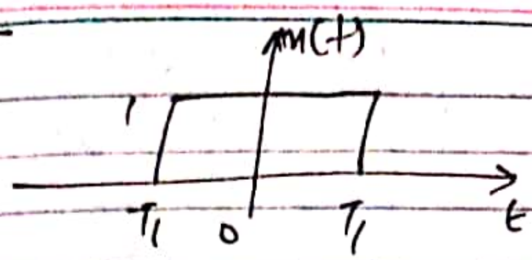

Spectrum → equal contribution of all frequencies

contains all frequencies.

$\boxed{x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)}$

## Lecture

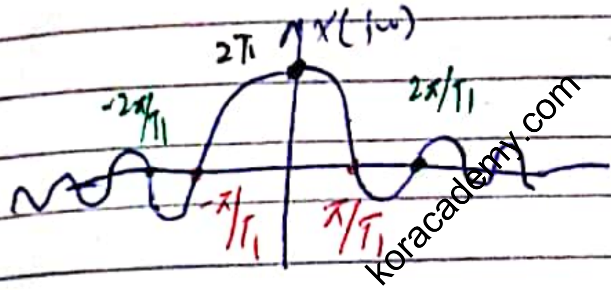
**Example**




$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-T_1}^{T_1} 1 \cdot e^{-j\omega t}\, dt$$

$$X(j\omega) = \frac{2 \sin \omega T_1}{\omega} \rightarrow \text{Similar to Sinc function.}$$

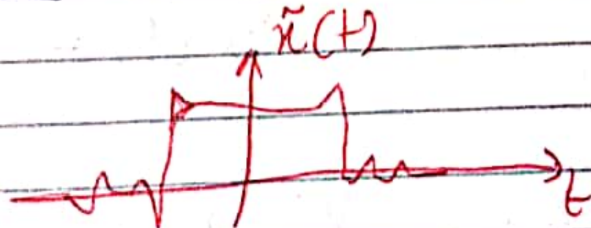



Or if going in reverse order i.e. from $X(j\omega)$ to $x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

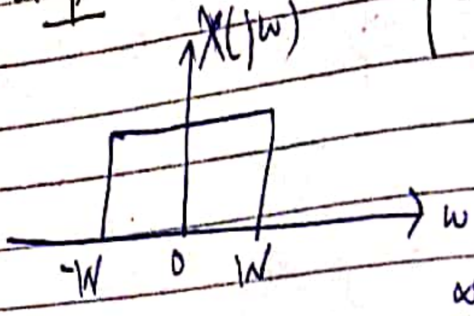
$$\tilde{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin \omega T_1}{\omega}\, e^{j\omega t}\, d\omega$$

Gibb's phenomena;




So for a finite duration signal, the bandwidth in frequency domain is infinite.

__Example__ $X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$



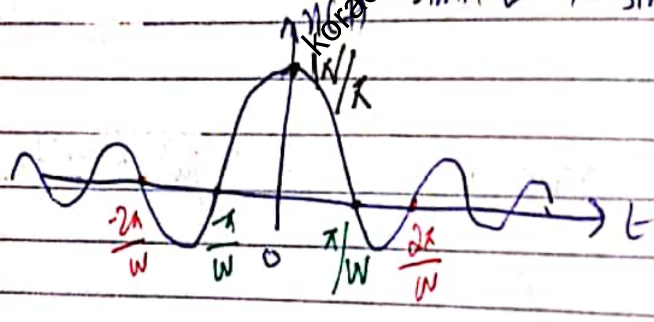

→ band limited signal

$x(t) = ?$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} 1\, e^{j\omega t}\, d\omega \Rightarrow x(t) = \frac{\sin Wt}{\pi t}$$

similar to sinc




__Duality.__ $x(t) \longleftrightarrow X(j\omega)$

$$\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$$

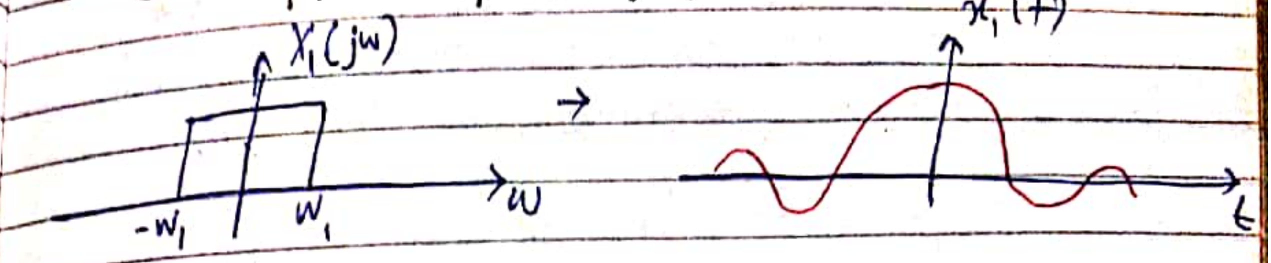
$X(j\omega) = 2T_1 \frac{\sin \omega T_1}{\omega T_1}$ $\qquad x(t) = W \frac{\sin Wt}{\pi t W}$

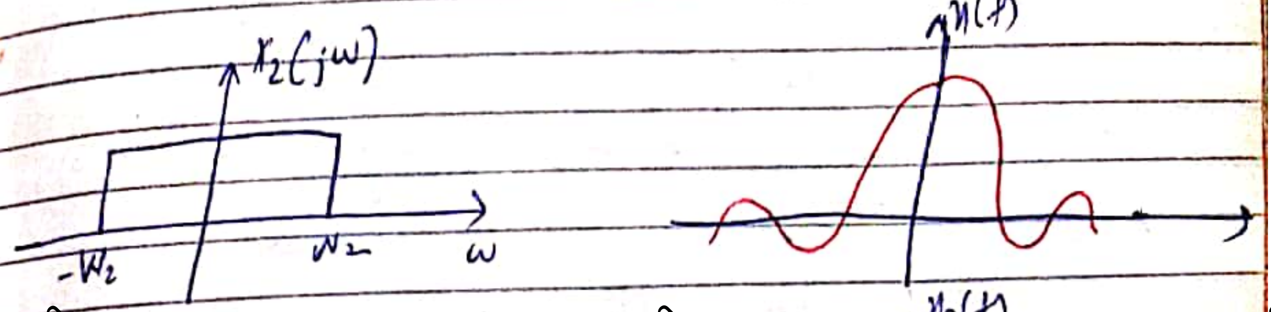
$X(j\omega) = 2T_1 \,\text{sinc}\left(\frac{\omega T_1}{\pi}\right)$ $\qquad x(t) = \frac{W}{\pi} \frac{\sin Wt}{Wt}$

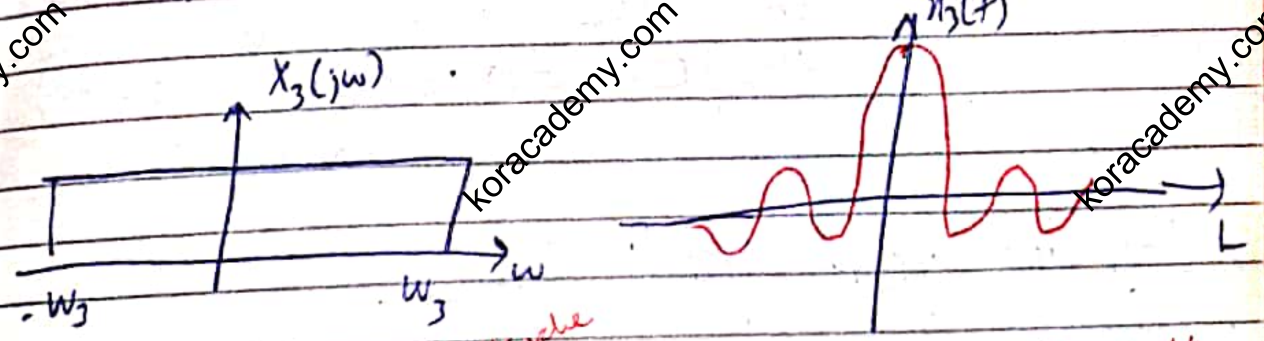
$$x(t) = \frac{W}{\pi}\,\text{sinc}\left(\frac{Wt}{\pi}\right)$$

→ in time domain scheme

## The effect of varying W. → (Missing band)



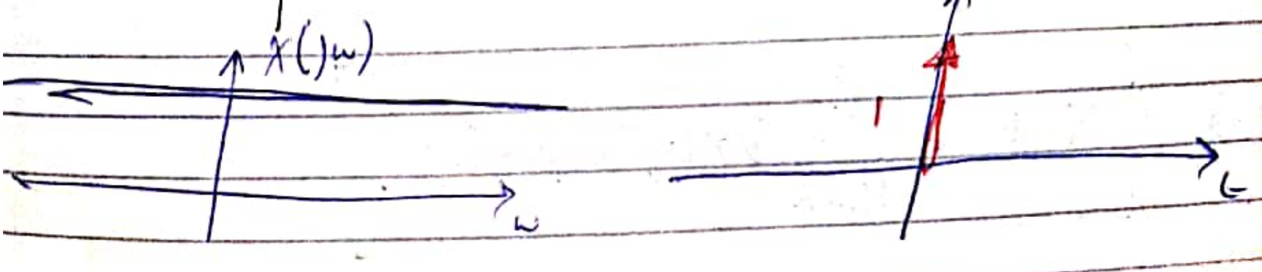

→ peak value

$W \uparrow \Rightarrow \frac{W}{\pi} \uparrow \Rightarrow \frac{\pi}{W} \downarrow$

↳ upward ↳ narrow

increasing bandwidth ⇒ signal is compressing in time domain and peak value at t=0 increases.

What if $W \to \infty$




Time and frequency domain are inversely related.

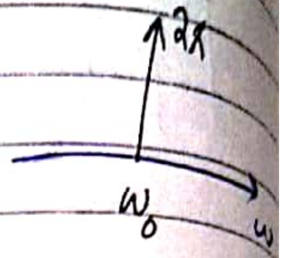
Bandwidth ↑ ⇒ signal compress.

↓ ⇒ expand.

# The Fourier Transform of Periodic Signals:

Assume that we have a fourier transform in the form of impulse function ie

$$X(j\omega) = 2\pi\,\delta(\omega-\omega_0)$$




$$x(t) = ?$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\,e^{j\omega t}\,d\omega = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \cancel{2\pi}\,\delta(\omega-\omega_0)\,e^{j\omega t}\,d\omega$$

$$x(t) = \int_{-\infty}^{\infty} \delta(\omega-\omega_0)\,e^{j\omega t}\,d\omega = e^{j\omega_0 t} \quad \text{(sifting property)}$$

↳ periodic signal

*

For any general periodic signal $x(t)$

FS $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x(t) = e^{j\omega_0 t} \xleftrightarrow{F} X(j\omega) = 2\pi\,\delta(\omega-\omega_0)$$

$$a_k e^{j\omega_0 t} \xleftrightarrow{F} 2\pi a_k\,\delta(\omega-\omega_0)$$

(: some term)

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\,\delta(\omega-k\omega_0)}$$

* There will always be impulse in FT of periodic signals.

$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$

**Example** $x(t) = \sin \omega_0 t$ → periodic

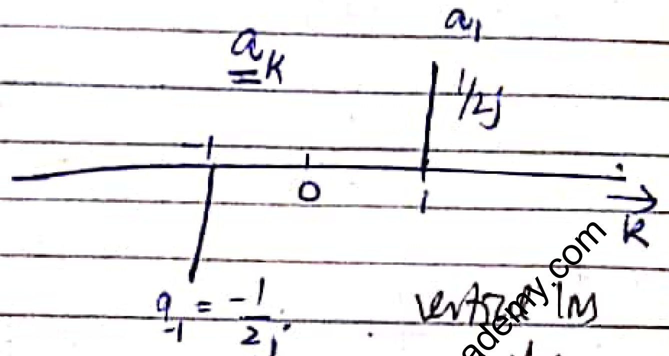
From Euler's relation;

$$x(t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$a_1 = \frac{1}{2j}, \quad a_{-1} = \frac{-1}{2j}$$

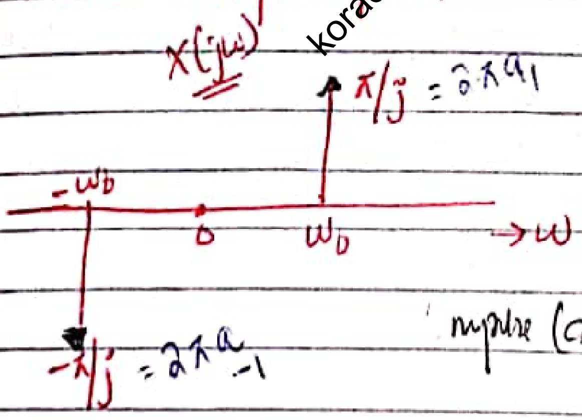
Fourier Series plot;

Line spectrum




vertical lines only

Fourier transform plot;




$(2\pi a_k)$

impulse (area)

If $a_2 \to 2\omega_0$

$a_3 \to 3\omega_0$ —

## Fourier Tool and LTI system

base signal (eigenvalue)

use for LTI system perspective

$$e^{j\omega t} \rightarrow \boxed{LTI} \rightarrow H(j\omega)\, e^{j\omega t}$$

$e^{j\omega t}$ ↓ eigen function

$H(j\omega)$ ↓ eigen value ↓ frequency response

$e^{j\omega t}$ → eigen function

linearity property

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \rightarrow \text{frequency response}$$

↳ fourier transform.

S/S Notes

Frequency response: It is basically the fourier transform of impulse response of LTI system

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow \boxed{LTI} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} H(jk\omega_0)$$

frequency response

now we have diff^t frequency ... what has LTI system done?

Suppose that an integral multiple of "$\omega_0$". It has changed the F.S.C

i/p F.Coeff $\rightarrow a_k$   o/p s/s for Coeff $= a_k H(jk\omega_0)$

↳ changing with the frequency response of the LTI system. (amplitude changes)

amplitude of F.C determines the amplitude of different freq (ω) inputs in the spectrum.

## Filtering

(specific freq)

① In some applications, it is required to change the relative amplitudes of the frequency components in a signal ② or ① to eliminate some frequency components entirely.

① Frequency shaping filters.
② Frequency selective filters.

where are practical systems where we see these filters?

## Frequency Shaping Filters

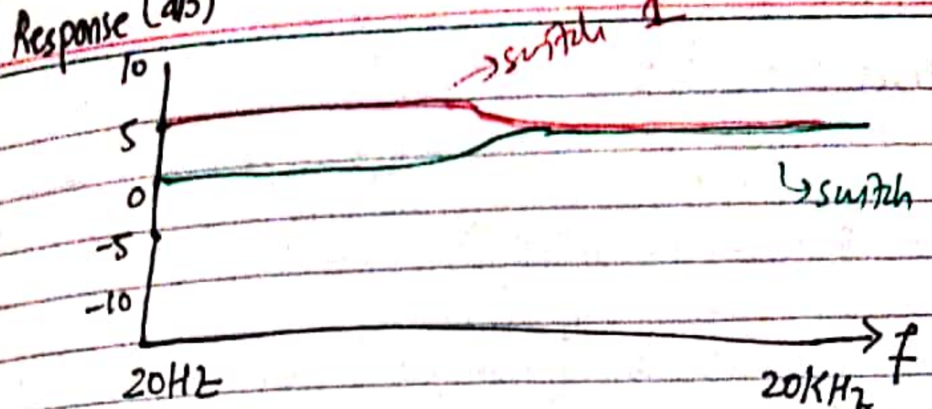
amplifiers

HiFi audio system: There are control buttons that allow listeners to change the relative amplitude of low frequency (Bass) and high frequency components (Treble) {Equalizer}

increasing B ⇒ ↑ L.F.C ⇒ heavy
↑ T ⇒ ↑ H.F.C ⇒ shrill

↳ laptop / computer sound settings.

+5 dB gain apply




changing relative amplitude (of some components) → sound quality

Increase amplitude of low frequency → heavy sound.

Increase amplitude of high frequency → shrill sound.

Practical / mathematical example of these filters.

Consider

differentiator — sinusoidal (periodic) signal

$x(t) \rightarrow \boxed{LTI} \rightarrow y(t) = \frac{d}{dt}x(t) = j\omega e^{j\omega t}$

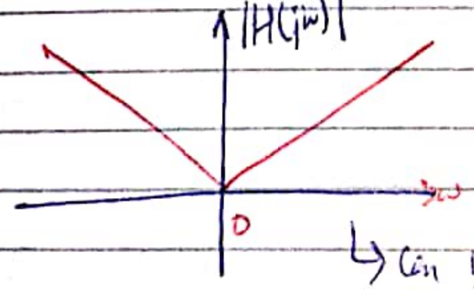
$x = e^{j\omega t}$

$j\omega$ → frequency response (eigen value)

↳ generally complex

→ $\boxed{H(j\omega) = j\omega}$

$|H(j\omega)| = |\omega|$ $\quad \measuredangle H(j\omega) = \pm \pi/2$ → depending on value of $\omega$.




→ (so it can be used as a high pass filter.)

↳ can be used to improve the higher frequencies in our signal.

High freq amplitude = ↑

Lower freq amplitude = ↓

→ This filter is useful in enhancing the rapid (higher frequencies) variations or transitions in a signal.

differentiator → emphasize H.F.s and deemphasize L.F.s

↳ in time domain → derivative

For in freq domain → H.p. filter

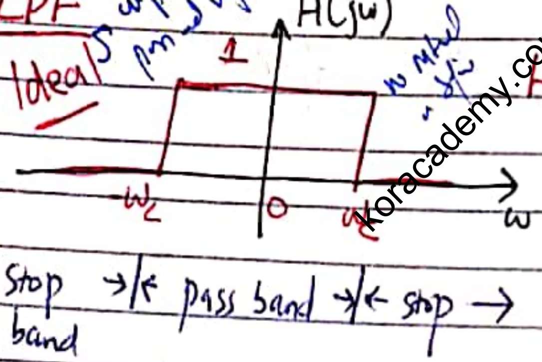
Different standard names → the basis of range of frequency that they pass / reject?

## Frequency selective Filters.

1. Low Pass filter. (LPF) → Passes low and attenuates high f.
2. High pass filter. (HPF) → passes high and rejects low f.
3. Band pass filter. (BPF) → passes a band & attenuates the rest.
4. Band stop filter (BSF) → stops a band and passes the rest.
   ↳ Notch filter

We will see which frequencies (type of filter) are being blocked and passed by frequency response.
↳ parameter name on the basis of which we are following of filtering.

LPF Ideal → completely pass and reject




$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

$\omega_c$ → cut off frequency.

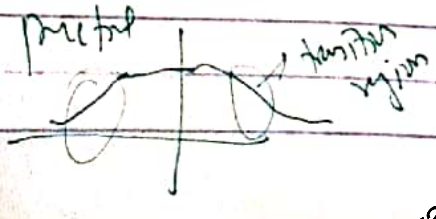
transition band

## Quality of frequency selective filters

1. How effective is the filter at passing frequencies in the pass band.
2. How effective is it at attenuating frequencies in the stop band.
3. How sharp is the transition near the cut off frequency.

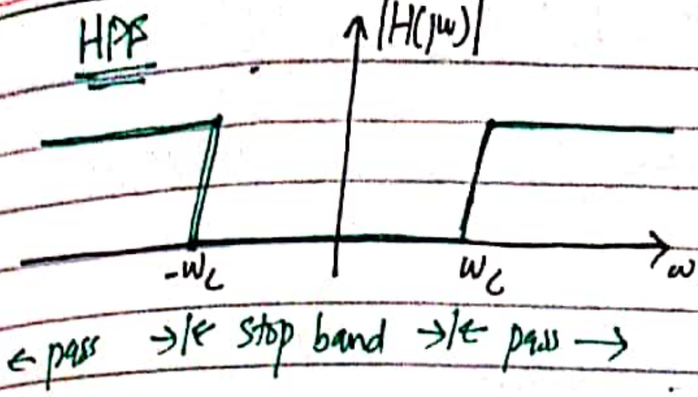
Ideal → infinite sharpness → during f pass to stop band.

practical




transition ↓ sharp → better (costly)

## HPF

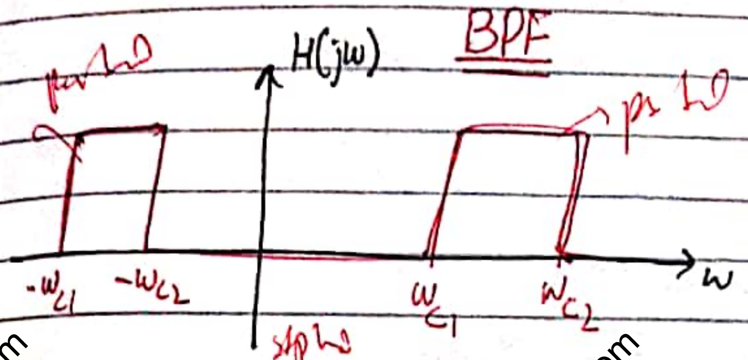



why am I calling these ideal?
why can we not design such set of filters practically?

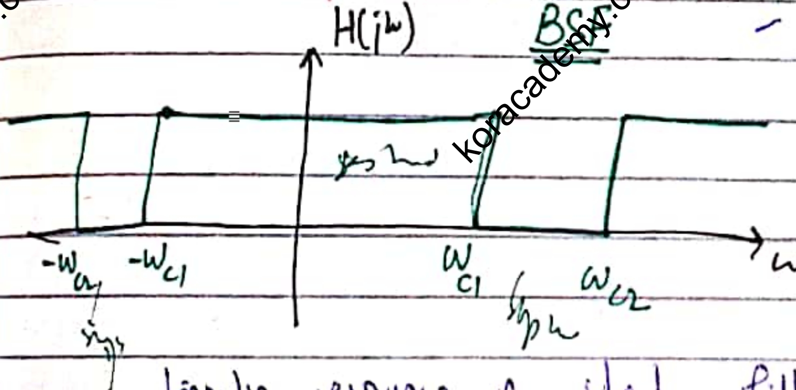
The shown graphs are frequency regions of the systems (filter).

The corresponding FT pair (h(t)) replace regions of fred;
h(t) ⇄ h[n]
n' sinc

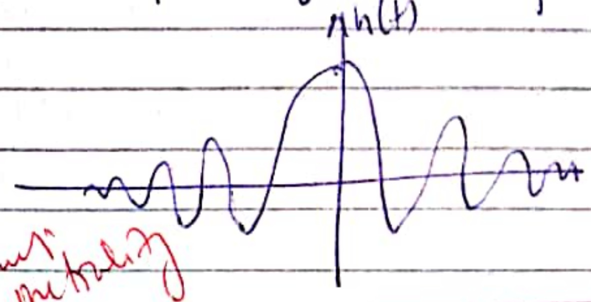
## BPF




## BSF




Impulse response of ideal filters; → sinc.




$\boxed{h(t) \neq 0 \text{ for } t<0 \;\downarrow\; \text{non causal}}$

problem with practicality

All practical systems are causal → so ideal filters are not possible

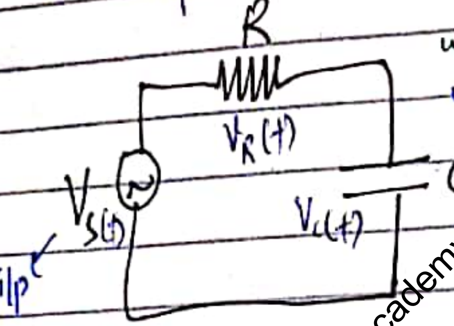
Cannot be physically realised.

Ideal filters discussed before are not realizable so we need to find filters that can be realized and then behavior / performance can be approximately close to ideal one.

## Realizable Filters.

depends on output comes what is the o/p

A simple RC circuit; can be used as a practical filter




what kind of filter is it? we might be interested in removing some frequencies.

Also represent $V_R(t)$ in terms of $V_S(t)$ and $V_C(t)$.

KVL ⇒

$$V_s(t) = RC\frac{dV_c}{dt} + V_c(t) \rightarrow \text{LCCDE} \rightarrow (A)$$

Linear constant coefficient differential equation

So this circuit is an LTI system. → Linear and time invariant

→ Frequency response = ? → only then can we characterize it and tell what sort of filter is it.

If sinusoidal i/p is applied;

ie $V_S(t) = e^{j\omega t}$

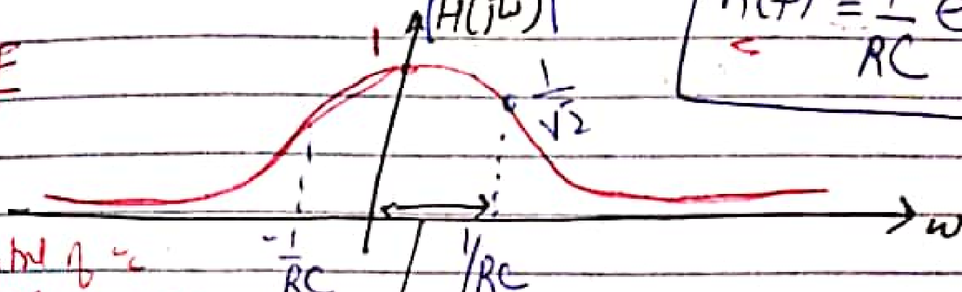
$$\rightarrow o/p = V_c(t) = H(j\omega)e^{j\omega t}$$

$$(A) \Rightarrow RC\frac{d}{dt}\{H(j\omega)e^{j\omega t}\} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$\Rightarrow RCj\omega\, H(j\omega)\, e^{j\omega t} + H(j\omega)\, e^{j\omega t} = e^{j\omega t}$$

$$\Rightarrow \boxed{H(j\omega) = \frac{1}{1 + RCj\omega}} \xleftrightarrow{IFT} \boxed{h(t) = \frac{1}{RC} e^{-t/RC}\, u(t)}$$

**LPF**

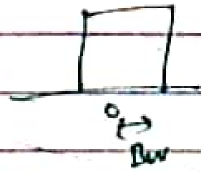



$\hookrightarrow = \omega_c$

BW (+ve)

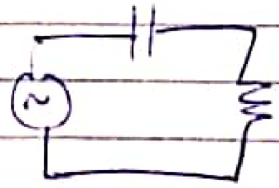
set in position of $\omega_c$ is not defined ⇒ a proper BW would not be defined. (The same point for BW ...)

$\hookrightarrow$ is the f at which filter o/p becomes $1/\sqrt{2}$ of the maximum.




no confusion in BW

⇒ Now If we use Resistor as o/p;



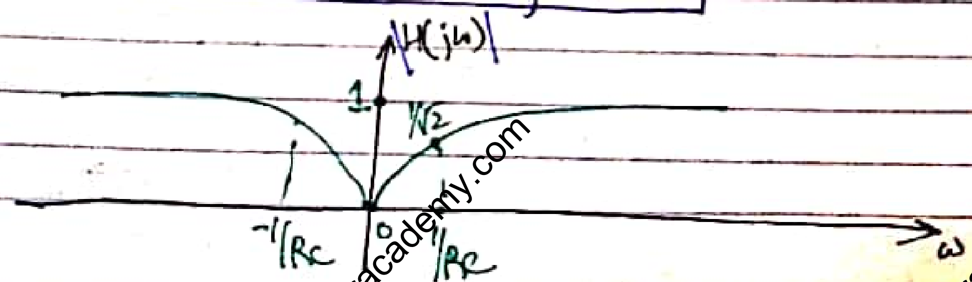
KVL and represent $V_c(t)$ in terms of $V_s(t)$ or $V_R(t)$.

$$RC\frac{d}{dt}V_R(t) + V_R(t) = RC\frac{d}{dt}V_s(t) \rightarrow (A)$$

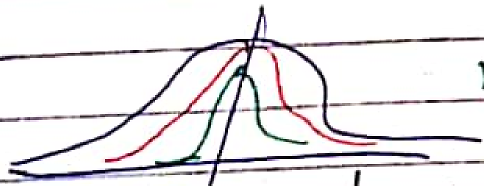
$$\left\{ \begin{array}{l} \text{If } V_s = e^{j\omega t} \\ V_R(t) = H(j\omega)\, e^{j\omega} \end{array} \right.$$

$$(A) \Rightarrow \boxed{H(j\omega) = \frac{j\omega RC}{1 + j\omega RC}}$$

**HPF**



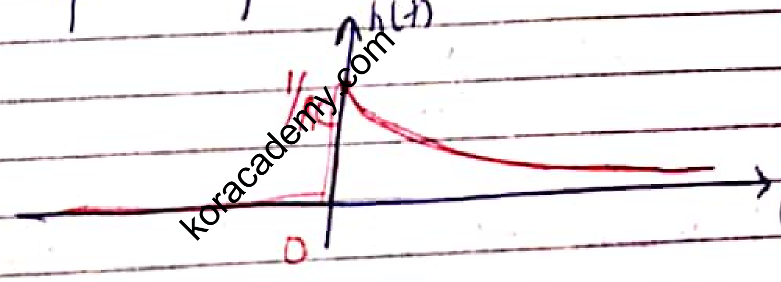
How to improve response of the filter?
i.e close to ideal.



Increase the number of resistors and capacitors.
cost ↑ complexity ↑

DSP → order of filter → Number of lumped components in the system

Impulse response of realizable LP RC filter;



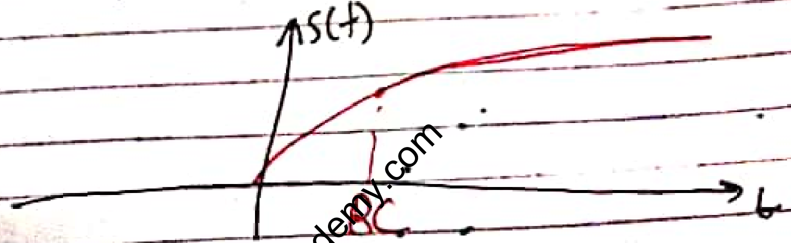

Realizable → the impulse response is zero for $t<0$
So causal.

## Step Response

$$u(t) \longrightarrow \boxed{LTI} \longrightarrow s(t)$$

$$h(t) = \frac{d\, s(t)}{dt} \quad \text{or} \quad s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$s(t) = \left(1 - e^{-t/RC}\right) u(t)$$




RC time constant

Limitations ⇒ when you analyze / design a filter, you would not conclude it to be a good or bad filter only on the basis of its frequency response. Also see impulse and step response (time domain)

→ As RC is increased, the frequency response improves but the time response becomes slower. ←

↳ circuit will operate slowly. performance when i/p is applied on the effect (o/p) will take some time to occur.

## Lecture

### Properties of CTFT.

analysis eq $\quad X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$

synthesis eq $\quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{+j\omega t} \, d\omega$

Notation $\quad \boxed{x(t) \xleftrightarrow{F} X(j\omega)}$ = fourier transform pair.

#### 1.- Linearity.

if $\quad x(t) \xleftrightarrow{F} X(j\omega)$

and $\quad y(t) \xleftrightarrow{F} Y(j\omega)$

Then

$$\boxed{a x(t) + b y(t) \xleftrightarrow{F} a X(j\omega) + b Y(j\omega)}$$

Proof. :

let $\quad z(t) = a x(t) + b y(t) \xleftrightarrow{F} Z(j\omega) = ?$

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} \, dt = \int_{-\infty}^{\infty} [a x(t) + b y(t)] e^{-j\omega t} \, dt$$

$$Z(j\omega) = a \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt + b \int_{-\infty}^{\infty} y(t) e^{-j\omega t} \, dt$$

$$\Rightarrow Z(j\omega) = a X(j\omega) + b Y(j\omega)$$

$$x(t+t_0) \xleftrightarrow{F} X'(j\omega) = e^{j\omega t_0} X(j\omega)$$ left shifting.

## 2. Time shifting.

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$\boxed{x(t-t_0) \xleftrightarrow{F} X'(j\omega) = e^{-j\omega t_0} X(j\omega)}$$ right ~~left~~ shift

As $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

so $X'(j\omega) = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$

let $t - t_0 = \tau \Rightarrow dt = d\tau$ → $t_0$ is constant

$$X'(j\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau+t_0)} d\tau = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} \cdot e^{-j\omega t_0} d\tau$$

$$X'(j\omega) = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau = e^{-j\omega t_0} X(j\omega)$$

$$|X(j\omega)| = |X'(j\omega)| \quad \text{as } |e^{-j\omega t_0}| = 1$$

So time shifting only effects the phase.

## 3. Conjuge and Conjugate Symmetry

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$\boxed{x^*(t) \xleftrightarrow{F} X^*(-j\omega)}$$

As $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega$$

Replace $\omega$ by $-\omega$

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(-j\omega)\, e^{j\omega t}\, d\omega$$

If $x(t)$ is a real signal, ie $x(t) = x^*(t)$

$\Rightarrow X(j\omega) = X^*(-j\omega)$

FT is conjugate symmetric

$X(j\omega) = |X(j\omega)| \measuredangle X(j\omega)$ a $X(j\omega) = Re\{X(j\omega)\} + Im\{X(j\omega)\}$

$|X(j\omega)| = |X(-j\omega)|$ even

and $\measuredangle X(j\omega) = -\measuredangle X(-j\omega)$ odd

a $Re\{X(j\omega)\} = Re\{X(-j\omega)\}$

$Im\{X(j\omega)\} = -Im\{X(-j\omega)\}$

If $x(t)$ is real and even;

then $X(j\omega)$ is real and even.

If $x(t)$ is real and odd;

then $X(j\omega)$ is imaginary and odd.

4.- Time Scaling: $x(t) \xleftrightarrow{F} X(j\omega)$

$$\boxed{x(at) \xleftrightarrow{F} X'(j\omega) = \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

a can be positive or negative

So $X'(j\omega) = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$

let $at = \tau \Rightarrow \frac{d\tau}{dt} = a$

$$\Rightarrow X'(j\omega) = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, \frac{d\tau}{a}$$

$X(j\frac{\omega}{a})$

$$\Rightarrow X'(j\omega) = \frac{1}{a} X\left(j\frac{\omega}{a}\right)$$

FT for which $\omega = \frac{\omega}{a}$

compress in time = expand in frequency and vice versa.

Indirect relation b/w time and frequency domain.

a multiplied in t domain and divided in f domain

## 5: Time Reversal

$x(t) \xleftrightarrow{F} X(j\omega)$

So by time scaling; $a = -1$,

$$\boxed{x(-t) \xleftrightarrow{F} X(-j\omega)}$$

## 6: Duality Property.

FT eq $\quad X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

both equations are similar although not identical

IFT eq $\quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

Rearranging $x(t)$ to get something like $X(j\omega)$

$$2\pi\, x(t) = \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

Replace $t$ by $-t$

$$2\pi . x(-t) = \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega t} d\omega$$

Replace $\omega$ by $t$ and $t$ by $\omega$.

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(jt) e^{-j\omega t} dt$$

$\uparrow$ $s = \sigma + j\omega$

$$\boxed{\begin{array}{l} x(t) \xleftrightarrow{F} X(j\omega) \\ X(t) \xleftrightarrow{F} 2\pi x(-\omega) \end{array}} \rightarrow \text{duality}$$

7. <u>Convolution Property</u>

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t) = x(t) * h(t)$$

If $x(t) \xleftrightarrow{F} X(j\omega)$

$y(t) \xleftrightarrow{F} Y(j\omega) = ?$

$h(t) \xleftrightarrow{F} H(j\omega)$

$$y(t) = x(t) * h(t) \longleftrightarrow Y(j\omega) = ?$$

As $Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} [x(t) * h(t)] e^{-j\omega t} dt$

$$Y(j\omega) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right] e^{-j\omega t} dt$$

Changing the order of integration;

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt\right] d\tau$$

↓ Time shift

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau} H(j\omega)\, d\tau$$

$$Y(j\omega) = H(j\omega) \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau}\, d\tau$$

$$\boxed{Y(j\omega) = H(j\omega)\, X(j\omega)}$$

$$\boxed{x_1(t) * x_2(t) \xleftrightarrow{F} X_1(j\omega)\, X_2(j\omega)}$$

frequency response

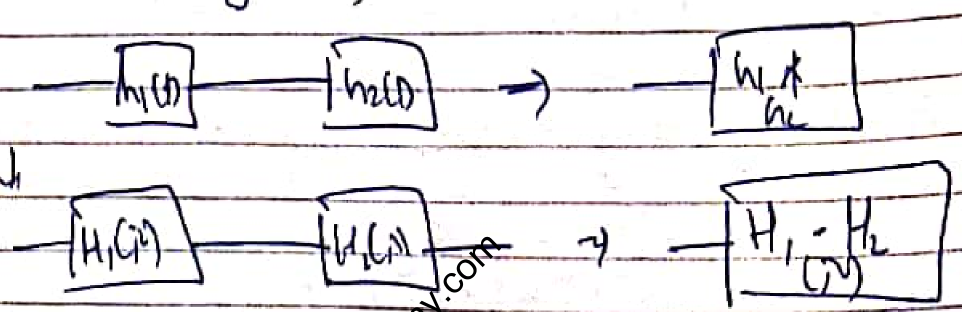
$$h(t) \xleftarrow{F^{-1}} H(j\omega) = \int_{-\infty}^{\infty} \underbrace{h(t)}_{\text{impulse response}} e^{-j\omega t}\, dt$$

Frequency response of the system does not exist if this integral will not converge

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$ stable system

↳ frequency response

In cascaded systems;



significance → in time domain → multiplying two signals (message) (carrier)

modulation → why? to shift frequency

convolution in f domain → information signal shifts to the frequency of carrier signal.

8. <u>Multiplication property</u>

$x_1(t) \xleftrightarrow{F} X_1(j\omega)$ $\quad$ $x_2(t) \xleftrightarrow{F} X_2(j\omega)$

then

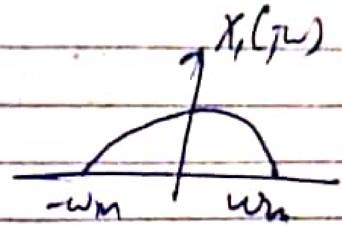
$$\boxed{\underset{\text{information}}{x_1(t)}\ \underset{\text{carrier}}{x_2(t)} \xleftrightarrow{F} [X_1(j\omega) * X_2(j\omega)] \frac{1}{2\pi}}$$

→ Also called modulation property. (in c.s)

convolution in frequency

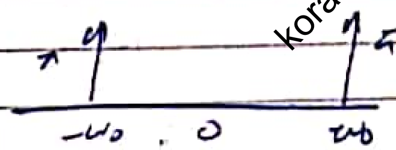
let $x_1(t) \cos\omega_0 t$

↳ carrier frequency




$x_1(t) \xleftrightarrow{F} X_1(j\omega)$

$\cos\omega_0 t \xleftrightarrow{F} \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$




$x(t)\cos\omega_0 t \xleftrightarrow{F} X_1(\omega) * \pi\delta(\omega-\omega_0)$

$+ X_1(\omega) * \pi\delta(\omega+\omega_0)$

$\pi X_1(\omega-\omega_0) + \pi X_1(\omega+\omega_0)$


9. <u>Differentiation Property</u>

<u>Time domain</u> $\quad x(t) \xleftrightarrow{F} X(j\omega)$

$$\boxed{\frac{d^n}{dt^n} x(t) \xleftrightarrow{F} (j\omega)^n X(j\omega)}$$

As $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$

$\therefore \frac{d}{dt} x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) j\omega e^{j\omega t} d\omega$

Frequency domain. $x(t) \xleftrightarrow{F} X(j\omega)$.

$$\boxed{t^K x(t) \xleftrightarrow{F} (j)^K \frac{d^K}{d\omega^K} X(j\omega)}$$

10. Frequency shift:

$x(t) \xleftrightarrow{F} X(j\omega)$

$$\boxed{e^{j\omega_0 t} x(t) \xleftrightarrow{F} X(j(\omega-\omega_0))}$$

(duality/modulation) carrier

11. Integration. $x(t) \xleftrightarrow{F} X(j\omega)$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)}$$

FT of constant term / average value / dc term.

Constant value $\rightarrow a_0 \xleftrightarrow{F} 2\pi a_0 \delta(\omega)$

Taking to LTI system

$x(t) \longrightarrow \boxed{\begin{array}{c}\text{Integrator}\\ h(t)=u(t)\end{array}} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

$= x(t) * h(t)$

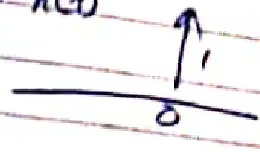
So. $y(t) = x(t) * u(t)$

$\hookrightarrow Y(j\omega) = X(j\omega)\, U(j\omega)$

$U(j\omega) = ?$ represent it as another function

$u(t) =$ say $x(t)$

$\delta(t) \xleftrightarrow{F} 1$ $\quad \frac{d}{dt} x(t)$



$$\frac{d}{dt} x(t) \xleftrightarrow{F} 1$$

I don't need FT of derivative rather I need of x(t).

$$j\omega X(j\omega) \xleftrightarrow{F} 1$$

no derivative, the avg value's effect zero

$$X(j\omega) \xleftrightarrow{F} \frac{1}{j\omega} + 2\pi\left(\frac{1}{2}\right)\delta(\omega)$$

↳ Avg value of u(t)

$$\Rightarrow U(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

→ exists only at $\omega = 0$

$$\Rightarrow Y(j\omega) = X(j\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$

$$Y(j\omega) = \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

## Parseval's Relation.

energy present in a signal can be related to the energy in FT of that signal.

$x(t) \rightleftarrows X(j\omega)$

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}$$

Proof

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt$$

$$\Rightarrow x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\, e^{-j\omega t}\, d\omega$$

$$= \int_{-\infty}^{\infty} x(t) \left[ \frac{1}{2\pi} \left[ \int_{-\infty}^{\infty} X^{*}(j\omega) e^{-j\omega t} d\omega \right\} dt \right].$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^{*}(j\omega) \left\{ \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^{*}(j\omega)\, X(j\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

## Systems Characterized by LCCDE: (Role of FT in solving such set of eqs).

$N^{th}$ order LCCDE: $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

[illegible]

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad \text{—— (A).}$$

Frequency response, $H(j\omega) = ?$

M1 If $x(t) = e^{j\omega t}$ then $y(t) = H(j\omega) e^{j\omega t}$

eigen function

Put these two values in (A)

M2 Take FT of (A)

$$\sum_{k=0}^{N} a_k\, \boxed{F\left\{ \frac{d^k y(t)}{dt^k} \right\}} = \sum_{k=0}^{M} b_k\, \boxed{F\left\{ \frac{d^k x(t)}{dt^k} \right\}}.$$

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$y(t) \xleftrightarrow{F} Y(j\omega)$$

Differentiation property:

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$\Rightarrow Y(j\omega) \sum_{k=0}^{N} a_k (j\omega)^k = X(j\omega) \sum_{k=0}^{M} b_k (j\omega)^k$$

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

$$\boxed{H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}}$$

We can find $H(j\omega)$ directly by looking at LCCDE coefficients.

Convolution property

$y(t) = x(t) * h(t)$

$Y(j\omega) = H(j\omega) X(j\omega)$

Example

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$b_0 = 1$, $k = 0$ $\Rightarrow$ $H(j\omega) = \dfrac{1}{2 + j\omega}$

$a_0 = 2$, $a_1 = 1$

$$\boxed{e^{-at} u(t) \overset{F}{\longleftrightarrow} \frac{1}{a + j\omega}}$$

$$h(t) = e^{-2t} u(t).$$

or by solving $h(t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} H(j\omega) e^{j\omega t} \, d\omega$

Exple $\frac{d^2}{dt^2}y(t) + 4\frac{dy(t)}{dt} + 3y(t) = \frac{d}{dt}x(t) + 2x(t)$

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

If $h(t) = ?$ → reduce the given fraction into sum simple terms.

By partial fraction expansion method

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3} = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}$$

$$H(j\omega) = \frac{A_1}{j\omega + 1} + \frac{A_2}{j\omega + 3} \qquad A_1 = A_2 = 1/2$$

$$H(j\omega) = \frac{1}{2}\left(\frac{1}{j\omega + 1}\right) + \frac{1}{2}\left(\frac{1}{j\omega + 3}\right)$$

$$\Rightarrow h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

Exple $x(t) = e^{-t}u(t)$ $\quad H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}$

$y(t) = ?$

$x(t) \longrightarrow$ [LTI $h(t)$ $H(j\omega)$] $\longrightarrow y(t) = ?$

$$y(t) = x(t) * h(t)$$

(1) $Y(j\omega) = X(j\omega)\, H(j\omega)$

$x(t) = e^{-t}u(t) \xleftrightarrow{F} \dfrac{1}{j\omega+1}$

$H(j\omega) = \dfrac{j\omega+2}{(j\omega+1)(j\omega+3)}$

$Y(j\omega) = \dfrac{j\omega+2}{(j\omega+1)^2(j\omega+3)}$

partial fraction

$Y(j\omega) = \dfrac{A_{11}}{j\omega+1} + \dfrac{A_{12}}{(j\omega+1)^2} + \dfrac{A_{21}}{j\omega+3}$

$A_{11} = \dfrac{1}{4},\ A_{12} = \dfrac{1}{2},\ A_{21} = \dfrac{-1}{4}.$

$Y(j\omega) = \dfrac{1}{4}\left(\dfrac{1}{j\omega+1}\right) + \dfrac{1}{2}\left(\dfrac{1}{(j\omega+1)^2}\right) + \dfrac{-1}{4}\left(\dfrac{1}{j\omega+3}\right)$

$\Rightarrow y(t) = \dfrac{1}{4}e^{-t}u(t) + \dfrac{1}{2}\, t\, e^{-t}u(t) - \dfrac{1}{4}e^{-3t}u(t)$ Ans

$$\boxed{\dfrac{t^{n-1}}{(n-1)!}\, e^{-at}u(t) \xleftrightarrow{F} \dfrac{1}{(a+j\omega)^n}}$$

$t\, e^{-at} u(t) \xleftrightarrow{} \dfrac{1}{(a+j\omega)^2}$

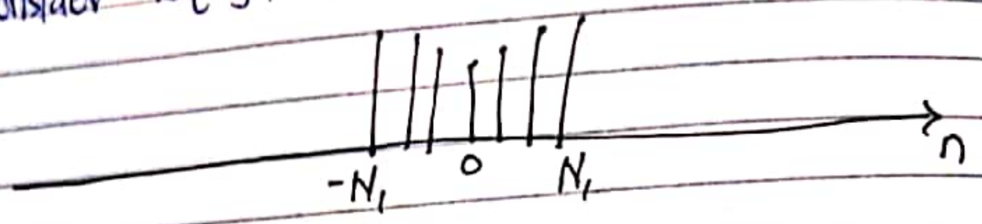
Lecture

# The Discrete Time Fourier Transform

→ used for Aperiodic signals.

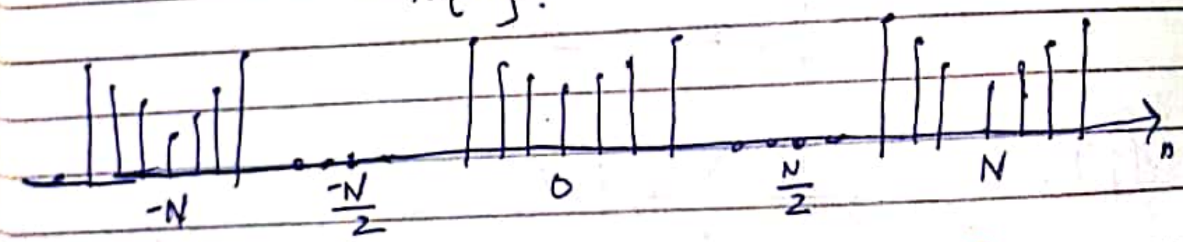
Aperiodic → periodic version → Increase period

consider $x[n]$,



We want to represent this Aperiodic in terms of complex exponentials i.e $e^{j\omega n}$.

Consider a periodic version of $x[n]$, $\tilde{x}[n]$.



$\tilde{x}[n]$ is a periodic version of $x[n]$.
$x[n]$ is one period of $\tilde{x}[n]$.

As $N$ increases, $x[n]$ and $\tilde{x}[n]$ become identical over a longer interval of time.

As $N \to \infty$, $x[n] = \tilde{x}[n]$ for all values of $N$.

For periodic signal $\tilde{x}[n]$, we have DTFS representation;

CTFS → summation infinite

DTFS → summation is over one period

↳ harmonically related signals are finite (repeat after $2\pi$) ...

$$\tilde{x}[n] = \sum_{K=\langle N\rangle} a_K e^{jK\left(\frac{2\pi}{N}\right)n}.$$ → Synthesis eq

$$a_K = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n] e^{-jK\left(\frac{2\pi}{N}\right)n}$$ → Analysis eq

we can consider any period.

↳ It is more convenient to use $-\frac{N}{2}$ to $\frac{N}{2}$.

$$a_K = \frac{1}{N}\sum_{n=-\frac{N}{2}}^{N/2} \tilde{x}[n] e^{-jK\left(\frac{2\pi}{N}\right)n}$$

the approximated signal is equal to the original one.

$$a_K = \frac{1}{N}\sum_{n=-\frac{N}{2}}^{N/2} x[n] e^{-jK\left(\frac{2\pi}{N}\right)n}$$

$$a_K = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n] e^{-jK\left(\frac{2\pi}{N}\right)n}$$ → $\omega_0$

or $$a_K = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n] e^{-jK\left(\frac{2\pi}{N}\right)n}$$

b/c function is zero outside $-N_1$ to $N_1$.

let $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$ → $K\omega_0$

↳ function of a continuous variable

$$\Rightarrow a_K = \frac{1}{N} X\left(e^{jK\omega_0}\right)$$

→ Fourier coefficients ($a_K$) are samples of $X(e^{j\omega})$ ←

$$\Rightarrow \tilde{x}[n] = \sum_{K=\langle N\rangle} \frac{1}{N} X\left(e^{jK\omega_0}\right) e^{jK\frac{2\pi}{N}n}.$$

As $\omega_0 = \frac{2\pi}{N} \Rightarrow \frac{\omega_0}{2\pi} = \frac{1}{N}$

F.S representation of the approximated signal.

$$\Rightarrow \boxed{\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0})\, e^{jk\left(\frac{2\pi}{N}\right)n}\, \omega_0}$$

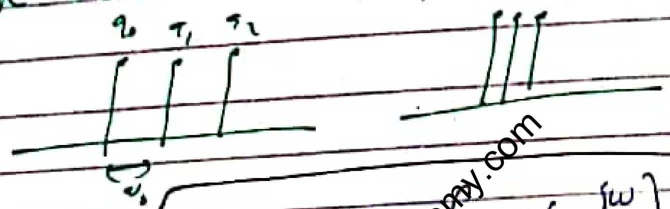
As $N \to \infty$, $\Rightarrow \tilde{x}[n] = x[n]$ $\Rightarrow \Sigma \to \int$ → integral $2\pi$

↳ $\omega_0 \downarrow$ $\Rightarrow \omega_0 \to 0$

↳ The fourier coefficients are closer to each other. (spacing decrease)

as $\omega_0$ has very small value

$k\omega_0 \to \omega$ (continuous)



Synthesis Eq

$$\boxed{x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega}$$

$2\pi$ → period

where

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}$$ → Analysis Eq

DTFS → finite sum

CTFS → infinite sum

Important point. Difference b/w CTFT and DTFT

As CTFT,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

and $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

① DTFT is always periodic with period $2\pi$.

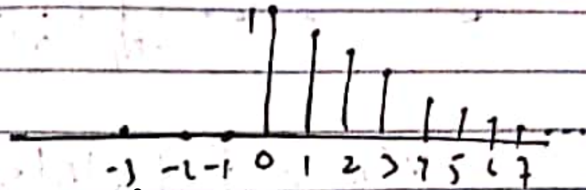
CTFT is nonperiodic

→ fundamental difference.

① The signal is a discrete time signal but its DTFT is continuous. (function of $\omega$)

**Example** $x[n] = a^n u[n]$, $|a| < 1$

If this condition is not given, the signal will not converge.




$X(e^{j\omega}) = ?$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$ → finite → for convergence

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}} \quad \text{Ans}$$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$

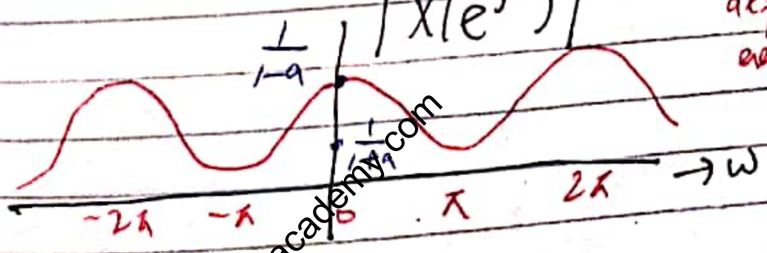
↓

Generally complex

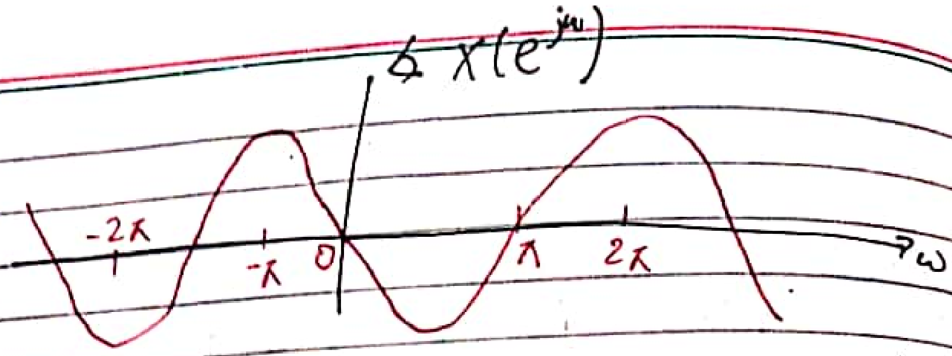
↳ mag + phase → value of a → +ve / −ve

$|X(e^{j\omega})|$, $\angle X(e^{j\omega})$ $\qquad e^{-j\pi} = -1$

If a is +ve $(0.9, 0.8 \ldots)$

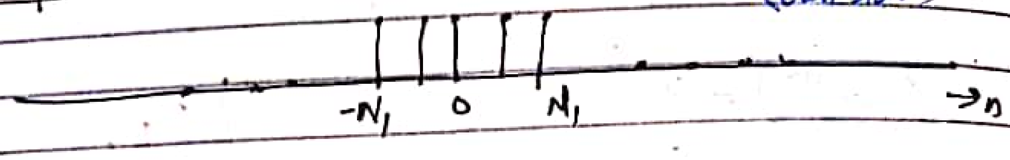



continuous and defined for any value of ω.




**Example** Rectangular pulse in DT.

(even signal)




$$x[n] = \begin{cases} 1, & |n| < N_1 \\ 0, & \text{otherwise} \end{cases}$$

real and even → real and even

As $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$

$\rightarrow X(e^{jw}) = \sum_{n=-N_1}^{N_1} 1\, e^{-jwn}$  DTFS

make lower limit zero.

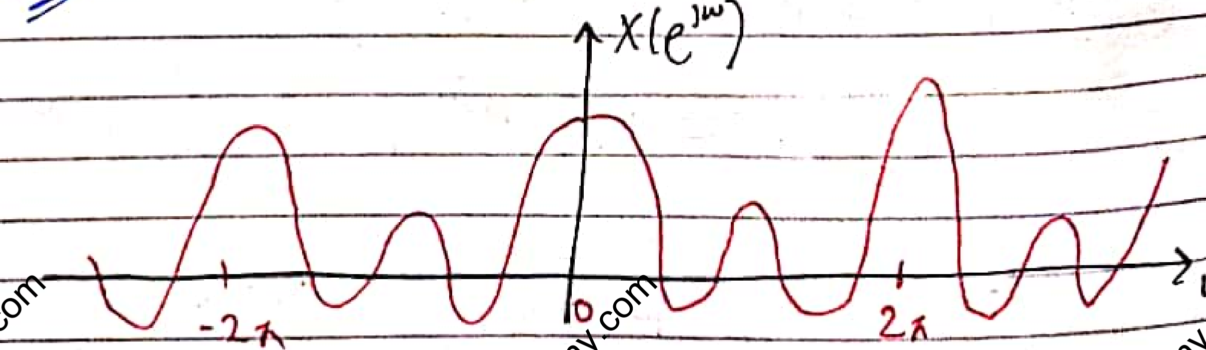
let $m = n + N_1$

$$X(e^{jw}) = \sum_{m=0}^{2N_1} e^{-jw(m-N_1)} = e^{jwN_1} \sum_{m=0}^{2N_1} (e^{-jw})^m$$

$$X(e^{jw}) = \frac{\sin w\left(N_1 + \frac{1}{2}\right)}{\sin(w/2)}$$

sinc

real



Another difference: CTFT [illegible] equation has issues of convergence as limits [illegible] $\infty$.

In DTFT analysis eq $X(e^{j\omega})$ has convergence issues as $\sum$ $n = -\infty$ to $\infty$ while synthesis eq $x(n)$ does not have [illegible].

## Convergence of DTFT:

If interval is not (-π, π) then 2π, the RHS will not be equally equal to the LHS.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

### Conditions

same as CTFT

1. The signal must be absolutely summable.

ie $$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

2. The signal must be finite energy signal,

ie $$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

### General Framework.

→ Fourier transform of periodic discrete time signals.

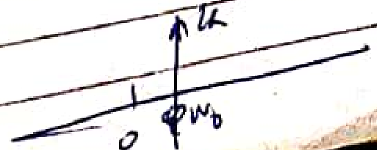
→ For periodic DT signals, the fourier transform must consist of impulses.

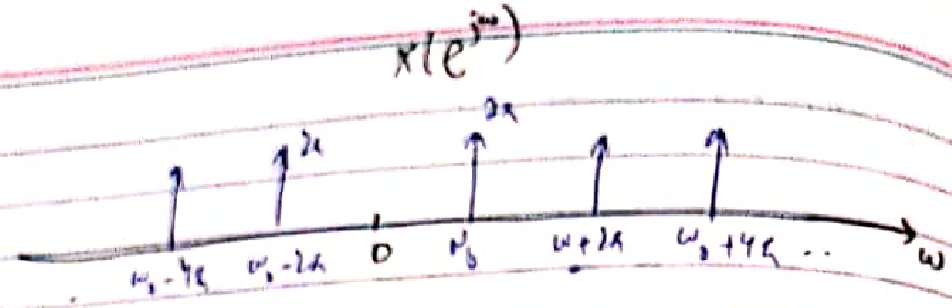
eg $x[n] = e^{j\omega_0 n}$. and it is periodic with period $2\pi$.

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi l)$$

As in CT: $x(t) = e^{j\omega_0 t}$

$$X(j\omega) = 2\pi\,\delta(\omega - \omega_0)$$





We assumed a specific periodic signal, but what if we have any general periodic signal?

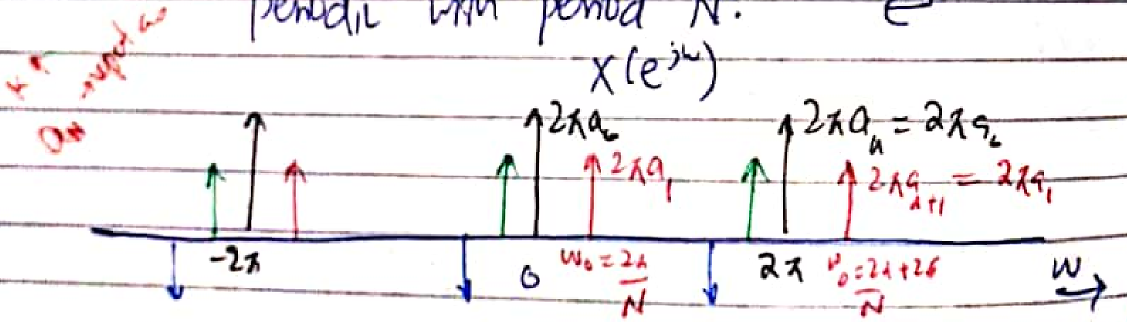
For any arbitrary periodic signal $x[n]$ with period $N$, fourier series exists.

ie $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$    $k\omega_0 \to$ harmonically related signals.

As $e^{j\omega_0 n} \xleftrightarrow{F} 2\pi\,\delta(\omega-\omega_0)$

$$\Rightarrow \boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta\left(\omega - \frac{2\pi}{N}k\right)}$$

periodicity already included due to $a_k$.
periodic with period $N$.



DTFT and filtering

$$x[n] = e^{j\omega n} \longrightarrow \boxed{\begin{matrix}\text{DT LTI}\\ h[n]\end{matrix}} \longrightarrow y[n] = H(e^{j\omega})\, e^{j\omega n}$$

eigen value

eigen function

$$\Rightarrow H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

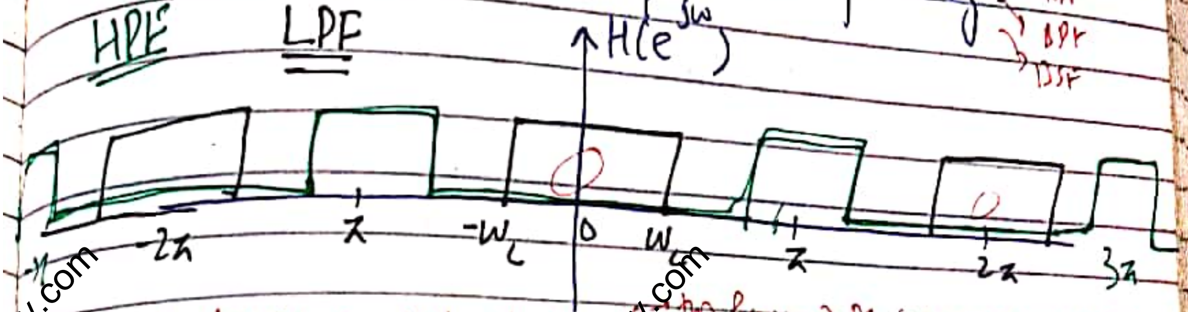
Eigen value (Frequency response) is the fourier transform of impulse response

For a CT LTI system, frequency response is Aperiodic

⇒ For a DT LTI system, frequency response is periodic. ⇐

Now we apply this concept to filtering → LPF, HPF, BPF, BSF

HPF LPF




In discrete domain, we told noticed that we this freqy is the same as 0. ch 2 in between 0 & 2π

frequencies near 0 and $2\pi$ are low frequencies.
frequencies near $\pi$ and $-\pi$ are high frequencies.

Example a system from

What type of filter is this?

$$x[n] \longrightarrow \boxed{\text{DT LTI}} \longrightarrow y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right)$$

So we find the frequency response, i.e. $H(e^{j\omega})$.

Impulse response, $h[n] = \frac{1}{2}\left(\delta[n] + \delta[n-1]\right)$.

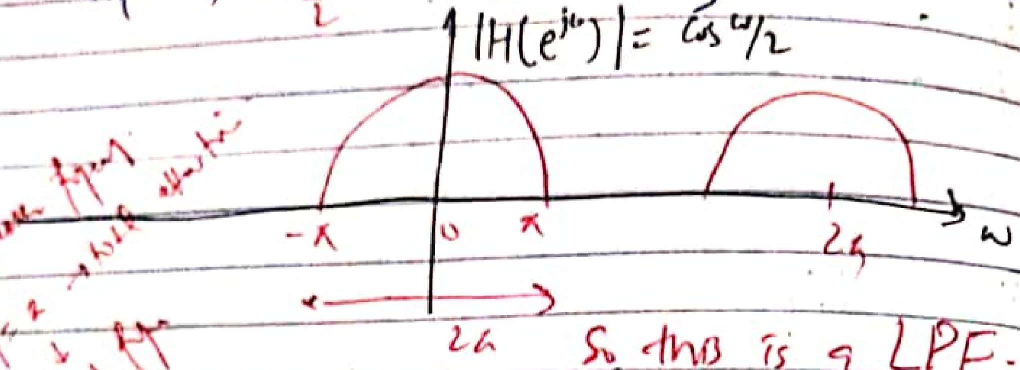
Now $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$

$= \frac{1}{2}\left(1 + e^{-j\omega}\right)$

$\Rightarrow H(e^{j\omega})$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{2} e^{-j\omega/2}\left(e^{j\omega/2} + e^{-j\omega/2}\right)$$

$$H(e^{j\omega}) = \frac{1}{2} e^{-j\omega/2}\left[2\cos(\omega/2)\right]$$




So this is a LPF.

o/p only depends on present i/p → Non recursive

o/p depends on present i/p and past o/p → Recursive

Consider $y[n] - a\,y[n-1] = x[n]$ (Recursive)

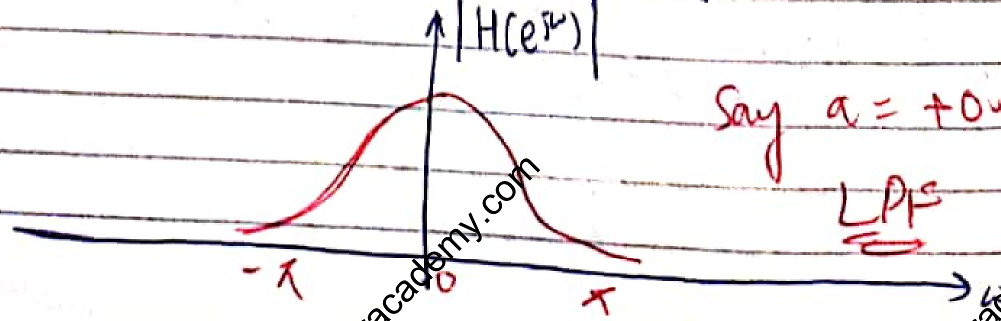
If $x[n] = e^{j\omega n}$ $\quad y[n] = H(e^{j\omega})e^{j\omega n}$

$$\Rightarrow H(e^{j\omega})e^{j\omega n} - a\,H(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$$

Rearrange for $H(e^{j\omega})$

$$\boxed{H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}}$$

$a = ?$




Say $a = +0.6$

LPF


the constant 'a' can effect property of filter

2. Consider a non causal DT filter; y[n] it only depend on i/p at x(n)

scaling / weights.

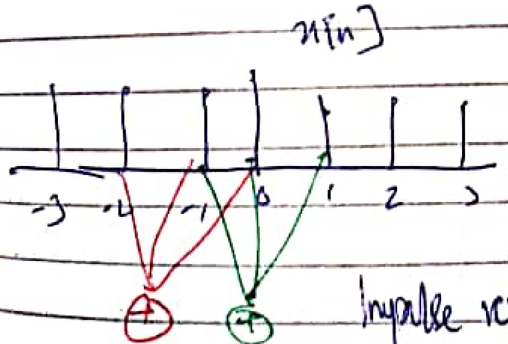
$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]$$

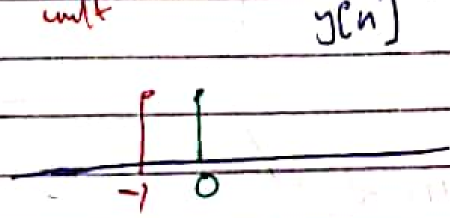
called a moving average filter

say we have a 3 point moving avg filter

the o/p is such that we are taking avg of three i/ps. (noncausally)

$$y[n] = \frac{1}{3}\left(\underbrace{x[n-1]}_{\text{prev}} + \underbrace{x[n]}_{\text{current}} + \underbrace{x[n+1]}_{\text{next}}\right)$$



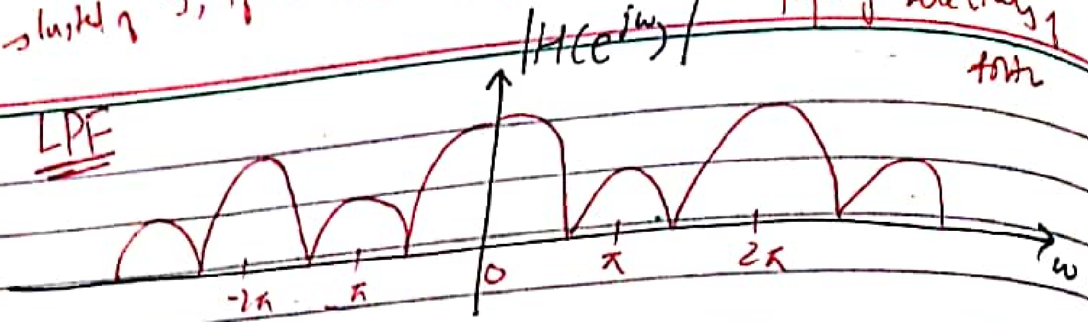

Impulse resp $h[n] = \frac{1}{3}\left[\delta(n-1) + \delta(n) + \delta(n+1)\right]$

$$H(e^{j\omega}) = \frac{1}{3}\left(e^{j\omega} + 1 + e^{-j\omega}\right)$$

$$\boxed{H(e^{j\omega}) = \frac{1}{3}\left(1 + 2\cos\omega\right)}$$

→ if compared to 3 point avg filter → filtering scheme has been narrowed.
→ instead of 3, if we take 4, 5, 6.. points, we can improve the frequency selectivity further.

## LPF




Instead of taking 3 points, take $N+M+1$ points moving average.

system taking avg of concrete i/ps, are passing in frequency and slowly higher ...

So the i/p relation for generalized moving point average filter will be;

3 pt: $\frac{1}{3}\sum_{-1}^{1} x[n-k]$

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]$$

→ moving avg

Op-amp filter not → frequency within (almost) reachable → slow varying data → size)

## Properties of DTFT

notation $\quad x[n] \xleftrightarrow{F} X(e^{j\omega}) \quad$ FT pair

### 1. Periodicity.

$$\boxed{X(e^{j(\omega+2\pi)}) = X(e^{j\omega})}$$

### 2. Linearity.

$$x_1[n] \xleftrightarrow{F} X_1(e^{j\omega}), \quad x_2[n] \xleftrightarrow{F} X_2(e^{j\omega})$$

$$\boxed{a x_1[n] + b x_2[n] \xleftrightarrow{F} a X_1(e^{j\omega}) + b X_2(e^{j\omega})}$$

### 3. Time shifting.

$$x[n] \xleftrightarrow{F} X(e^{j\omega})$$

$$\boxed{x[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0} X(e^{j\omega})}$$

4. Frequency Shift $x[n] \xleftrightarrow{F} X(e^{j\omega})$

$$\boxed{e^{j\omega_0 n} x[n] \xleftrightarrow{F} X(e^{j(\omega-\omega_0)})}$$

carrier information → spectrum shifts modulation when you multiply.

5. Difference Property. $x[n] \xleftrightarrow{F} X(e^{j\omega})$

$$x[n] - x[n-1] \xleftrightarrow{F} X(e^{j\omega}) - e^{-j\omega}(X(e^{j\Omega}))$$

$$\boxed{x[n] - x[n-1] \xleftrightarrow{F} X(e^{j\omega})[1 - e^{-j\omega}]}$$

6. Summation / Acumlation

$x[n] \xleftrightarrow{F} X(e^{j\omega})$ $\pi X(0)\delta(\omega)$

For constt / DC / exponential

$$\boxed{\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{F} \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi X(e^{j0})\delta(\omega - 2\pi k)}$$

y/c constt value is periodic

Duality property.

Convolution property $x[n] \rightarrow \boxed{\begin{matrix}\text{LTI} \\ h(n)\end{matrix}} \rightarrow y(n)$
$= x(n) * h(n)$.

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

multiplication

$$x(n) \cdot y(n) \longleftrightarrow X(e^{j\omega}) * Y(e^{j\omega})$$

Continuous time

Discrete time

**Fourier Series**

[No duality]

• Time domain

$$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Continuous time
Periodic in time

Frequency Domain

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t}$$

Discrete frequency
Aperiodic in frequency

Time domain

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

Discrete time
Periodic in time

Frequency domain

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

Duality exists

Discrete frequency
Periodic frequency

**Fourier Transform**

Duality exists.

Time domain

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Aperiodic in time
continuous time

Frequency domain

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Aperiodic in frequency
Continuous in frequency

duality

Time domain

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Discrete in time
Aperiodic in time

Frequency domain

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

No duality

Continuous in frequency
Periodic in frequency

FS

Lecture

## Application of DTFT on systems characterized by LCCDE

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{DT} \\ \text{LTI}\end{matrix}} \longrightarrow y[n].$$

i/p and o/p represented by $N^{th}$ order LCCDE;

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

Frequency response, $(H e^{j\omega}) = ?$

$$H(e^{j\omega}) \overset{F}{\longleftrightarrow} h[n]$$

**M1** For $x[n] = e^{j\omega n}$

$$\Rightarrow y[n] = H(e^{j\omega})\, e^{j\omega n}.$$

Substitute $x[n]$ and $y[n]$ into LCCDE and solve for $H(e^{j\omega})$.

**M2** Take DTFT of LCCDE;

$$F\left\{\sum_{k=0}^{N} a_k\, y[n-k]\right\} = F\left\{\sum_{k=0}^{M} b_k\, x[n-k]\right\}$$

$$\Rightarrow \sum_{k=0}^{N} a_k F\{y[n-k]\} = \sum_{k=0}^{M} b_k\, F\{x[n-k]\}$$

k Tosh pyp-1j

$$\Rightarrow \sum_{k=0}^{N} a_k (e^{-j\omega})^{k} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k (e^{-j\omega})^{k} X(e^{j\omega})$$

$$\Rightarrow Y(e^{j\omega}) \sum_{k=0}^{N} a_k e^{-j\omega k} = X(e^{j\omega}) \sum_{k=0}^{M} b_k e^{-j\omega k}$$

$$\Rightarrow \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

$$\Rightarrow \boxed{H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}}$$

<u>Example</u>

$$y[n] - a\,y[n-1] = x[n].$$

Find $H(e^{j\omega})$ and $h[n]$.

$$H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

We know $h[n] = \frac{1}{2\pi} \int_{2\pi} H(e^{j\omega}) e^{j\omega n} d\omega$

Basic DTFT pairs;

$$a^n u[n] \xleftrightarrow{F} \frac{1}{1 - a e^{-j\omega}}$$

$$\rightarrow h[n] = a^n u[n].$$

ANS

**Example** $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$

$H(e^{j\omega})$ and $h[n] = ?$

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

2nd order polynomial as prod of 1st order

Using partial fraction: to transform $H(e^{j\omega})$ in terms of 1st order fns.

$$H(e^{j\omega}) = \frac{2}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)}$$

$$= \frac{A}{\left(1 - \frac{1}{2}e^{-j\omega}\right)} + \frac{B}{\left(1 - \frac{1}{4}e^{-j\omega}\right)}$$

Solving $\Rightarrow$ $A = 4,\ B = -2$

$$\Rightarrow H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j2\omega}}$$

$$\Rightarrow 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n] = h[n]$$

Ans

# The Laplace Transform :

① → Laplace transform is the generalization of CTFT.

② → Laplace transform is the CTFT of a signal multiplied with real exponential signals.

(The generalization of DTFT is z transform)

let's go into the mathematical aspect of Laplace

$$x(t) = e^{st} \longrightarrow \boxed{\begin{matrix} LTI \\ h(t) \end{matrix}} \longrightarrow y(t) = H(s)\, e^{st}$$

where $s$ = complex $s = \sigma + j\omega$ sigma

$\boxed{\begin{matrix} \text{In CTFT} \\ \sigma = 0 \\ \text{also } e^{j\omega t} \end{matrix}}$

whether $s$ is complex, real or imaginary, it is eigen function of LTI system.
→ complex exponential

$H(s)$ → Transfer function @ system function.
→ eigen value in matrix

Convolution → $y(t) = x(t) * h(t)$

$$\boxed{H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt}$$ → Laplace transform

system function is the Laplace transform of impulse response $h(t)$.

→ when $s$ is purely imaginary, system function reduces to frequency response. ←

Generalizing for any signal; $x(t)$.

LT of $x(t)$; $$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt}$$

In pair form

$$\boxed{x(t) \xleftrightarrow{L} X(s)}$$

① ⇒ CTFT is the special case of LT.

$$\underset{s = j\omega}{\text{CTFT}} \qquad \underset{s = \sigma + j\omega}{\text{LT}}$$

For $s = j\omega$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

↳ CTFT

$$\boxed{X(s)\Big|_{s=j\omega} = F\{x(t)\}}$$

LT evaluated at $s = j\omega$ same FT of that signal

② What is general relation b/w LT and F other than $s = j\omega$.

If $s = \sigma + j\omega$

As $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

Laplace is a form of FT

$$\Rightarrow X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$$

FT of this signal

Laplace

behaviour in time

$$\rightarrow \boxed{X(s) = F\{x(t)\, e^{-\sigma t}\}}$$

→ Decay if $\sigma$ is +ve.
grow if $\sigma$ = -ve

Example $\quad x(t) = e^{-at} u(t)$

$$\Downarrow$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-at} . u(t)\, dt$$

This will converge only for $a > 0$;

$$X(j\omega) = \frac{1}{j\omega + a}, \quad a > 0$$

Now $\quad X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) . e^{-st}\, dt$$

$$X(\sigma + j\omega) = \int_{0}^{\infty} e^{-(\sigma+a)t} . e^{-j\omega t}\, dt$$

By relating FT & LT and finding FT

Will converge only if $\sigma + a > 0$

$$X(\sigma + j\omega) = \frac{1}{(\sigma+a) + j\omega}, \quad \sigma + a > 0 \Rightarrow \sigma > -a,$$

$$\rightarrow \boxed{X(s) = \frac{1}{s+a}, \quad Re\{s\} > -a}$$

FT of any signal does not converge

Laplace transform may converge for some values of $Re\{s\}$ and not for others.



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For $a>0$;

$$X(s) = \frac{1}{s+a}, \quad Re\{s\} > -a$$

$$X(s)\Big|_{s=j\omega} = \frac{1}{j\omega + a} \qquad \text{CTFT}$$

For $a<0$

CTFT does not converge ⇒ will not have a FT.

LT ?

LT still exists | converges because the $\boxed{Re\{s\} > -a}$ irrespective of a being +ve or -ve.

There are signals whose CTFT does not exist but LT does.

Example $x(t) = -e^{-at}u(-t)$, $X(s) = ?$

As $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = -e^{-at}e^{-st} =$

$$\boxed{X(s) = \frac{1}{s+a} \qquad Re\{s\} < -a}$$

There can be different functions with the same laplace transform (with different notations)

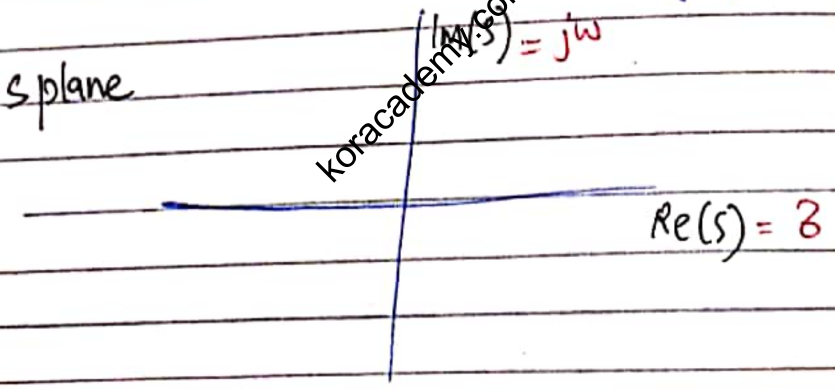
There are two things associated with laplace transform; completely characterized by two things.

① Algebric expression. $X(s) =$ ___

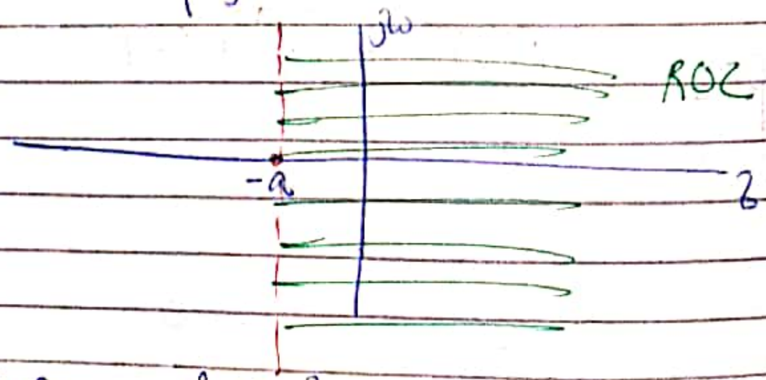
② Values of $s$ for which this algebric expression is valid. (integral should be finite.)

↳ Region of convergence (ROC)

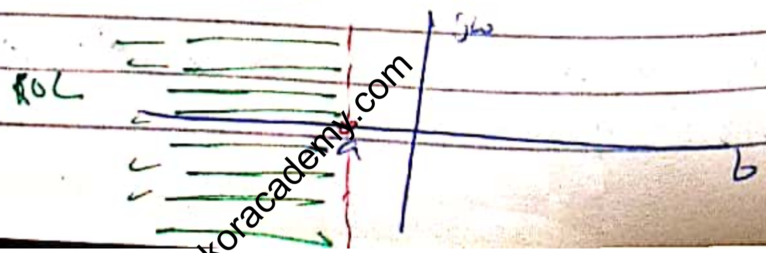
↳ we will also show it graphically




As in example 1;
$Re\{s\} > -a$ or $\sigma > -a$




In Ex 2; $Re\{s\} < -a$; or $\sigma < -a$



linear combination of the signals,

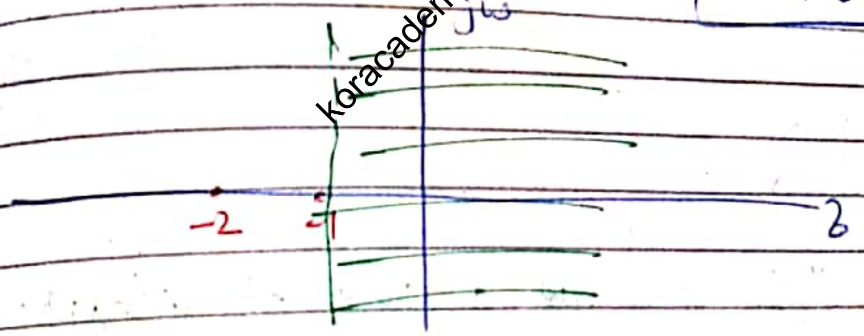
Example $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} \left(3e^{-2t}u(t) - 2e^{-t}u(t)\right) e^{-st} dt$

$\Rightarrow X(s) = 3\int_{0}^{\infty} e^{-2t} e^{-st} dt - 2\int_{0}^{\infty} e^{-t} e^{-st} dt$

$\Rightarrow X(s) = 3 \dfrac{1}{\underbrace{s+2}_{>-2}} - 2 \cdot \dfrac{1}{\underbrace{s+1}_{>-1}}$.

$\Rightarrow$ ROC; $\boxed{Re\{s\} > -1}$




→ Most of the time we will get LT in the form,

$X(s) = \dfrac{N(s)}{D(s)}$ → Rational LT.

Roots of $N(s)$ → zeros of $X(s)$, $X(s) = 0$ → o

Roots of $D(s)$ → poles of $X(s)$, $X(s) = \infty$ → x

$X(s) = \dfrac{1}{s+2}$

$s = -2$ → root of $D(s)$ ⇒ pole

Example $X(s) = \frac{s-1}{s+2}$ ROC $Re\{s\} > -2$

$s = 1 \Rightarrow$ root of $N(s) \to$ zero of $X(s)$

$s = -2 \to$ root of $D(s) \to$ pole of $X(s)$


## Poles and zeros at Infinity

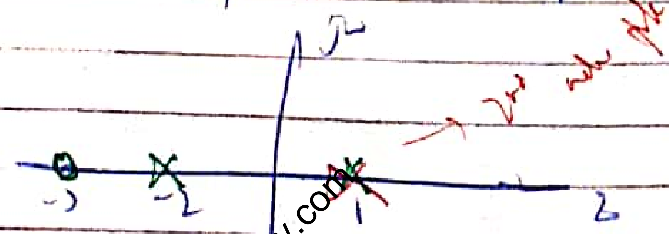
→ If the order of $D(s)$ is greater than the order of $N(s)$, then $X(s)$ will have zero at infinity.

→ If order of $N(s)$ is greater than order of $D(s)$, then $X(s)$ will have a pole at infinity.

Eg
$$X(s) = \frac{s+3}{(s-1)^2(s+2)}$$

$s = -3 \to$ zero $\quad s = -2$ pole

$(s+1)^2 \Rightarrow$ two poles at $s = 1$ (order of zero & pole)



# Properties of ROC.

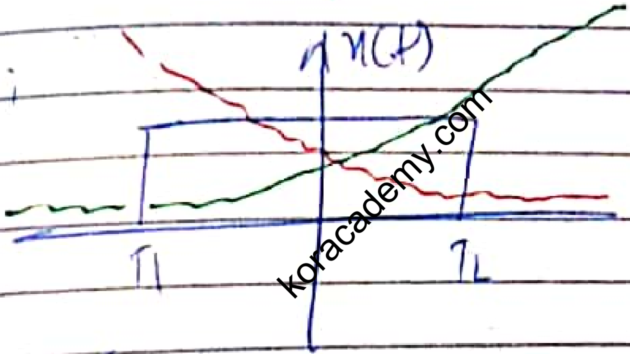
Re{s}

① The ROC of X(s) consists of strips parallel to the jω axis in the s plane.

② For rational LT X(s), the ROC does not contain any poles. → LT will not converge.

③ If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s plane.

consider:




$$\int_{-\infty}^{\infty} |x(t)| < \infty$$

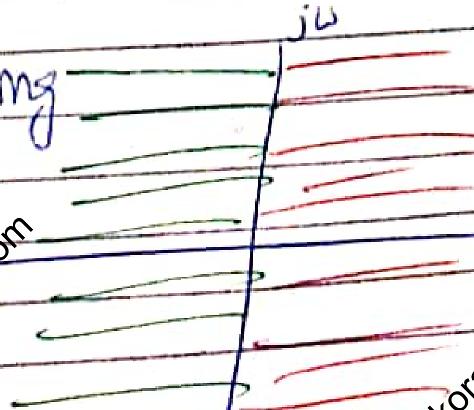
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$$

For what values of σ will it converge

ie $X(s) < \infty$.

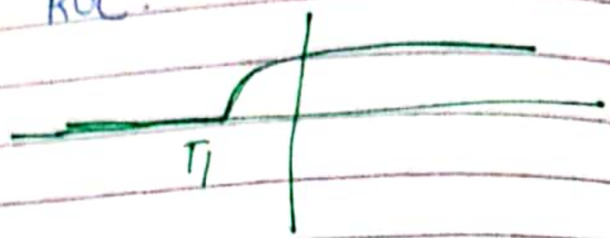
$$e^{-\sigma t} \begin{cases} \boxed{\sigma > 0} \rightarrow \text{decaying} \\ \boxed{\sigma < 0} \rightarrow \text{growing} \end{cases}$$

entire s plane is ROC.



(4) If $x(t)$ is right sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of $s$ for which $Re\{s\} > \sigma_0$ will also be in the ROC.

zero till $T_1$ and non zero afterwards.





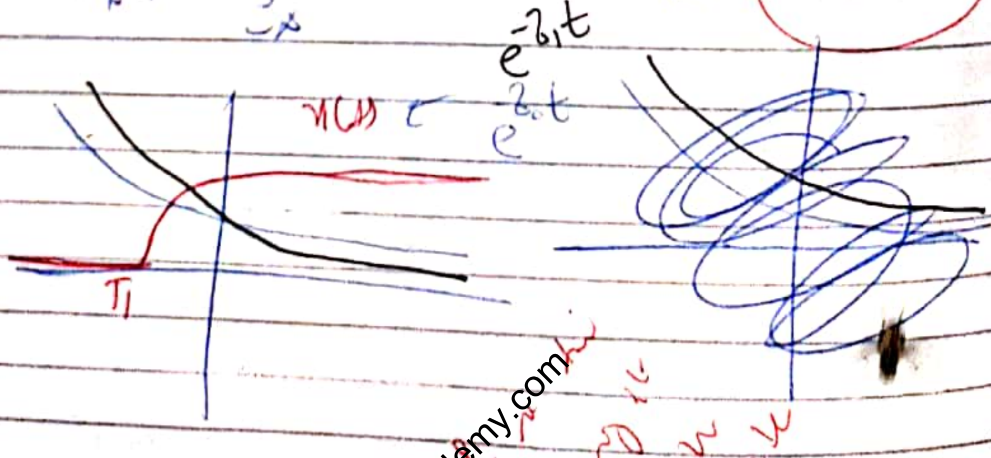
So ROC is also right sided for right sided signal.

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt < \infty$$

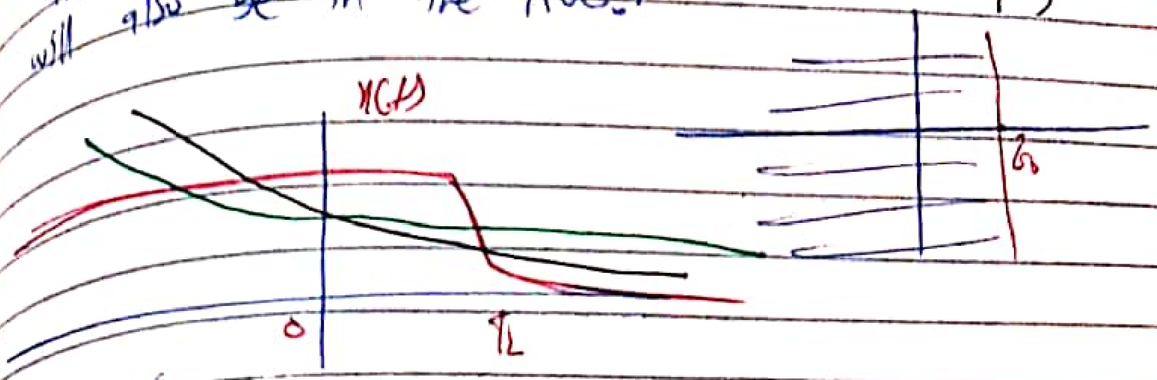
let $\sigma = \sigma_0$ $\quad X(s) = \int_{-\infty}^{\infty} x(t) e^{-\sigma_0 t} e^{-j\omega t} dt$

let $\sigma_1 > \sigma_0$ so corresponding signal is $e^{\sigma_1 t}$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-\sigma_1 t} e^{-j\omega t} dt \quad < \infty$$



(5) If $x(t)$ is a left sided signal and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all the values of $s$ for which $Re\{s\} < \sigma_0$ will also be in the ROC.



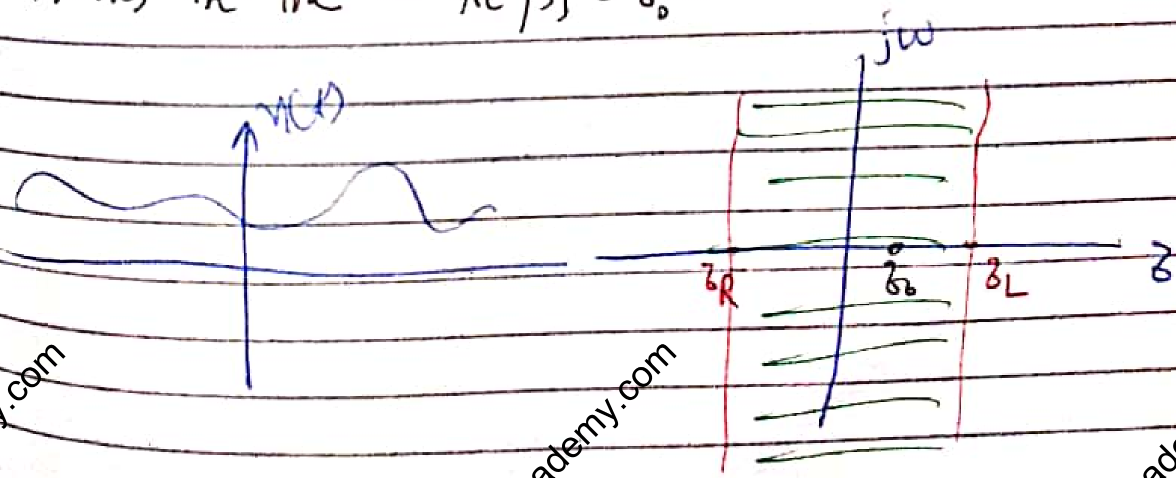

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma_0 t}\, e^{-j\omega t}\, dt < \infty$$

for $\sigma_1 < \sigma_0$

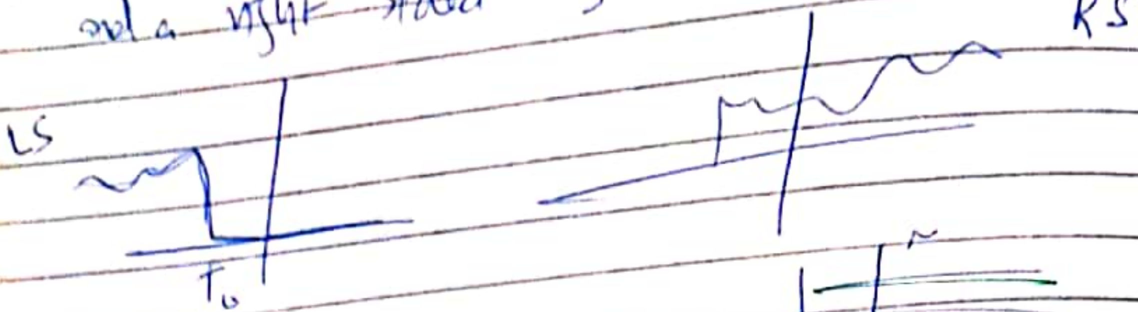
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma_1 t}\, e^{-j\omega t}\, dt < \infty$$

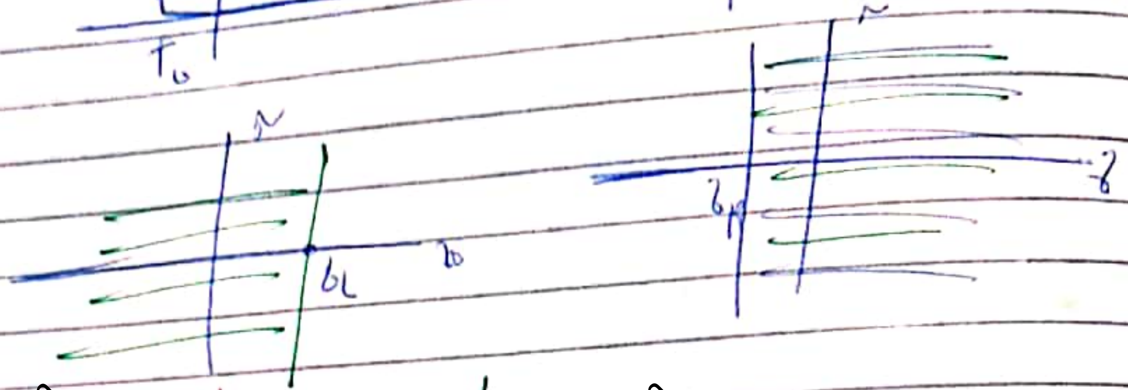
$e^{\sigma_0 t}$, $e^{\sigma_1 t}$ exp

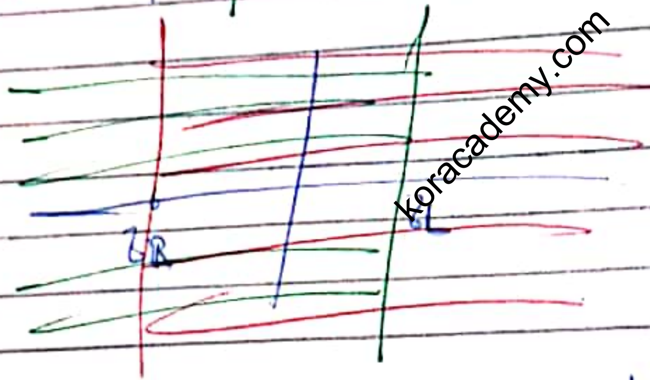
(6) If $x(t)$ is two sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC then the ROC will consist of a strip in the s plane that includes the line $Re\{s\} = \sigma_0$



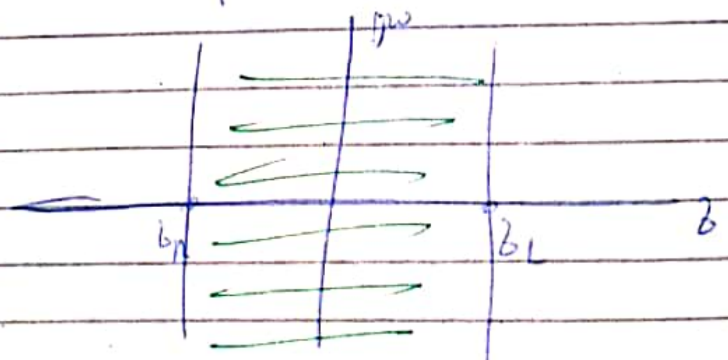
Divide the signal into two parts i.e a left and a right sided signal;




→ The ROC for two sided $x(t)$ is;



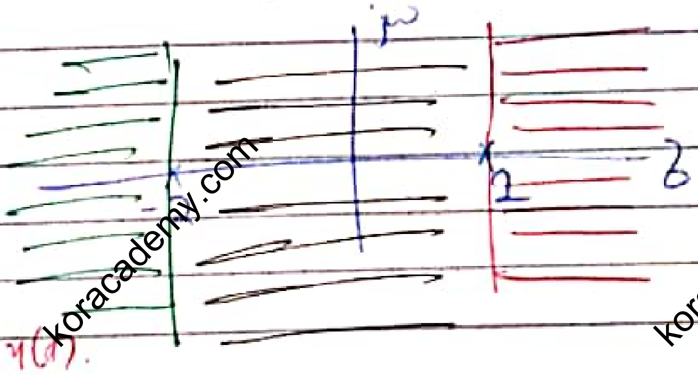

(*) If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition no poles of $X(s)$ are contained in the ROC.

eg in $\frac{1}{s+2}$ $Re\{s\} > -2$

⑧ If $X(s)$ is rational, then if $x(t)$ is right sided, the ROC is to the right of the right most pole.
If $x(t)$ is left sided then ROC is to the left of left most pole.

Say $X(s) = \dfrac{1}{(s-1)(s+2)}$

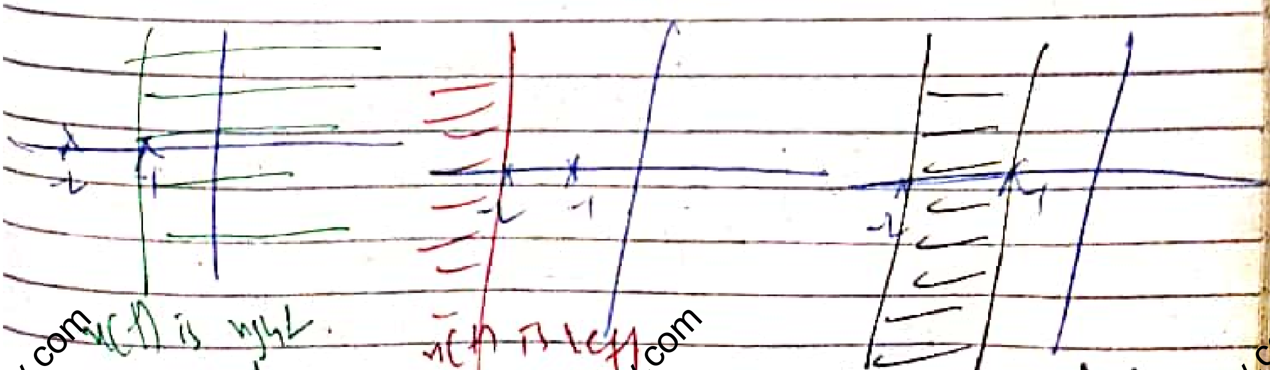
$x(t)$ is ~~right sided~~




If right sided $x(t)$.
If $x(t)$ is left sided..
If $x(t)$ is two sided.

__Example:__ $X(s) = \dfrac{1}{(s+1)(s+2)}$

poles $= -1, -2$ Possible ROC $= ?$




$x(t)$ is right sided

$x(t)$ is left sided

$x(t)$ is two sided

# Lecture

The same system expression can be related with three different ROCs.

ROC may give us information about x(t).

We haven't discussed the synthesis eq.

Laplace Tr. $\boxed{X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt}$ Analysis Eq

Inverse Laplace transform $x(t) = \{X(s)\}$ ?

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$$

FT of $x(t)\, e^{-\sigma t}$

$\boxed{\text{IFT eq}} \Rightarrow x(t)\, e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{j\omega t}\, d\omega$

Multiply b.s by $e^{\sigma t}$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{(\sigma + j\omega)t}\, d\omega$$

we need the values of $\sigma$ in the ROC

$s = \sigma + j\omega$ wrt $\omega$

$ds = j\, d\omega \Rightarrow d\omega = \frac{ds}{j}$

$$\boxed{x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds}$$

Synthesis equation

difficult / time consuming

complex integration

$X(s) \longleftrightarrow x(t)$ ILT

$\hookrightarrow$ partial fraction

expansion

Example ① $X(s) = \frac{1}{(s+1)(s+2)}$ $\quad Re\{s\} > -1$

$x(t) = ?$

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

solving $\Rightarrow$ $A = 1,\ B = -1$ individually right sided

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

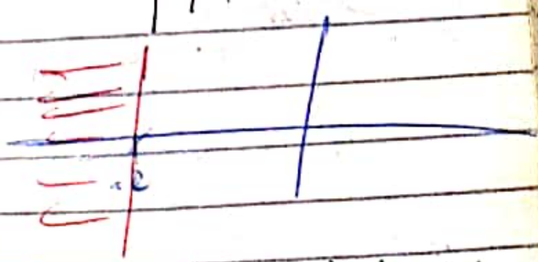
$$\boxed{e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}}$$

$$e^{-t}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+1}, \quad Re\{s\} > -1$$

$$e^{-2t}u(t) \overset{L}{\longleftrightarrow} \frac{1}{s+2}, \quad Re\{s\} > -2$$

$$\Rightarrow \boxed{x(t) = e^{-t}u(t) - e^{-2t}u(t)}$$

Example ② $X(s) = \frac{1}{(s+1)(s+2)}$ $\quad Re\{s\} < -2$



Similarly by partial fraction

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$ individually left sided

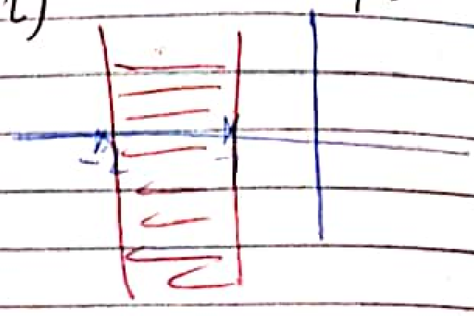
$$-e^{-t}u(-t) \overset{L}{\longleftrightarrow} \frac{1}{s+1} \quad Re\{s\} < -1$$

$$-e^{-2t}u(-t) \overset{L}{\longleftrightarrow} \frac{1}{s+2} \quad Re\{s\} < -2$$

$-e^{-at}u(-t) \longleftrightarrow \frac{1}{s+a}$

$$\Rightarrow \boxed{x(t) = -e^{-t}u(-t) + e^{-2t}u(-t)}$$

Exple (5) $X(s) = \frac{1}{(s+1)(s+2)}$, $-2 < Re\{s\} < -1$



Again partial fraction;

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$-e^{-t}u(-t) \longleftrightarrow \frac{1}{s+1}$, $Re\{s\} < -1$ L-S

$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$, $Re\{s\} > -2$ R-S

$$\Rightarrow \boxed{x(t) = -e^{-t}u(-t) + e^{-2t}u(t)}$$

## Properties of the Laplace Transform:

(1) **Linearity**

$x_1(t) \xleftrightarrow{L} X_1(s)$ with ROC $= R_1$

$x_2(t) \xleftrightarrow{L} X_2(s)$ with ROC $= R_2$

$$\boxed{a x_1(t) + b x_2(t) \xleftrightarrow{L} a X_1(s) + b X_2(s)}$$

with ROC containing $R_1 \cap R_2$ at least.

maybe more than intersection.

**Example**

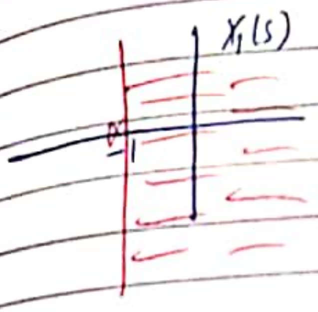
$$x(t) = x_1(t) - x_2(t)$$

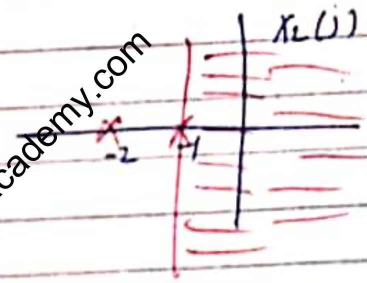
$$X_1(s) = \frac{1}{s+1} \quad Re\{s\} > -1$$

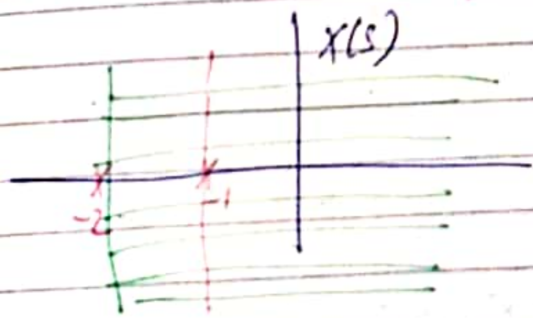
$$X_2(s) = \frac{1}{(s+1)(s+2)} \quad Re\{s\} > -1$$

Applying linearity property $X(s) = X_1(s) - X_2(s)$

$$X(s) = \frac{1}{(s+1)} - \frac{1}{(s+1)(s+2)} = \frac{1}{s+2}$$



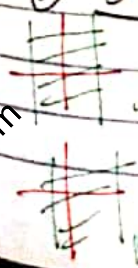

If ∩ of ROC = ∅ & LT doesn't exist

## (2) Time shifting Property

$$x(t) \overset{L}{\longleftrightarrow} X(s) \quad \text{with } ROC = R$$

$$\boxed{x(t \mp t_0) \overset{L}{\longleftrightarrow} e^{\mp st_0} X(s)} \quad \text{with } ROC = R$$

## (3) Shifting in s-domain



$$x(t) \overset{L}{\longleftrightarrow} X(s) \quad \text{with } ROC = R$$

$$\boxed{e^{s_0 t} x(t) \overset{L}{\longleftrightarrow} X(s - s_0)} \quad \text{with } ROC = R + Re\{s_0\}$$

$R + Re\{s_0\}$ $s_0 = +ve$ shift

④ **Time Scaling:**

$$x(t) \overset{L}{\longleftrightarrow} X(s) \quad \text{with ROC} = R$$

$$\boxed{x(at) \overset{L}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right)} \quad \text{with ROC} = aR$$

$a<1$ compress
$a>1$ expand

⑤ **Time reversal.**

Special case of time scaling with $a=-1$.

$$\boxed{x(-t) \overset{L}{\longleftrightarrow} X(-s)} \quad \text{with ROC} = -R$$

⑥ **Convolution property**

$$x_1(t) \overset{L}{\longleftrightarrow} X_1(s) \quad \text{with ROC} = R_1$$

$$x_2(t) \overset{L}{\longleftrightarrow} X_2(s) \quad \text{with ROC} = R_2$$

$$\boxed{x_1(t) * x_2(t) \overset{L}{\longleftrightarrow} X_1(s) \cdot X_2(s)}$$

with ROC containing at least $R_1 \cap R_2$

⑦ **Conjugation property**

$$x(t) \overset{L}{\longleftrightarrow} X(s) \quad \text{with ROC} = R$$

$$\boxed{x^*(t) \overset{L}{\longleftrightarrow} X^*(s^*)} \quad \text{with ROC} = R$$

If $x(t)$ is real $\Rightarrow x(t) = x^*(t)$

$\hookrightarrow X(s) = X^*(s^*)$

## ⑧ Differentiation Property

**In Time domain:**

$$x(t) \xleftrightarrow{L} X(s) \quad \text{with ROC} = R$$

$$\boxed{\frac{d^k}{dt^k} x(t) \xleftrightarrow{L} s^k X(s)}$$ with ROC at least containing R.

If y pole at $\frac{1}{s} \rightarrow s=0 \rightarrow$ removes pole, ROC increases.

**In s domain**

$$x(t) \xleftrightarrow{L} X(s) \quad \text{with ROC} = R$$

$$\boxed{-t\,x(t) \xleftrightarrow{L} \frac{d X(s)}{ds}}$$ with ROC = R

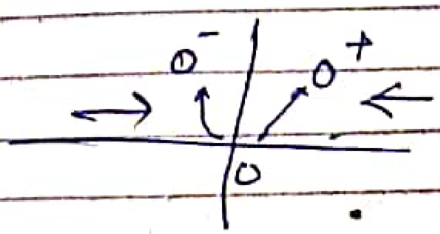
## Initial and Final value Theorems

→ If $x(t) = 0$ for $t<0$ and

$x(t)$ contains no impulses or higher order singularities (discontinuities) at the origin.

$$x(0^-) = \lim_{s \to \infty} s X(s)$$

Initial value theorem.




→ If $x(t) = 0$ for $t<0$
and $x(t)$ has a finite limit at $t \to \infty$.

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s X(s)$$

↳ final value theorem.

# Basic Laplace transform Pairs (Table 9.2)

| $x(t)$ | $H(s)$ | ROC |
|---|---|---|
| Impulse $\delta(t)$ | $1$ | Entire s plane |
| $u(t)$ | $1/s$ | $Re\{s\} > 0$ |
| $-u(-t)$ | $1/s$ | $Re\{s\} < 0$ |
| $\frac{t^{n-1}}{(n-1)!}u(t)$ | $1/s^n$ | $Re\{s\} > 0$ |

## Analysis and characterization of LTI systems using the Laplace transform.

$$x(t) \rightarrow \boxed{\begin{matrix} LTI \\ h(t) \end{matrix}} \rightarrow y(t) = x(t) * h(t)$$

$$X(s) \rightarrow \boxed{\begin{matrix} LTI \\ H(s) \end{matrix}} \rightarrow Y(s) = H(s)\, X(s)$$

system function

ROC = ?

Causality and stability with impulse response?

$\rightarrow H(s)$ $\rightarrow h(t)$ System function

## Causality

In time domain,

for causal LTI system;

$$h(t) = 0 \quad \text{for } t < 0.$$

↳ right sided signal.

$h(t) \leftrightarrow H(s)$ = — ROC = right sided half plane

"The ROC associated with the system function for a causal system is a right half plane."

The converse of this statement is not necessarily true. i.e (If the ROC is right half plane, it is a causal system) ✗

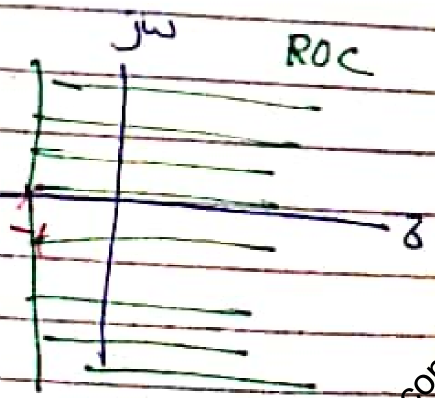
It guarantees only that the impulse response is right sided.


when will this converse statement be true?

→ For a system with a rational system function causality of the system is equivalent to the ROC being the right half plane to the right of the right most pole.

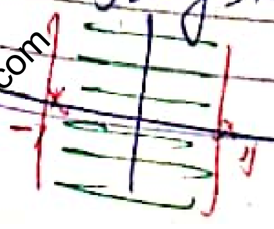
Example $h(t) = e^{-t}u(t)$
→ system is causal

$$H(s) = \frac{1}{s+1}$$

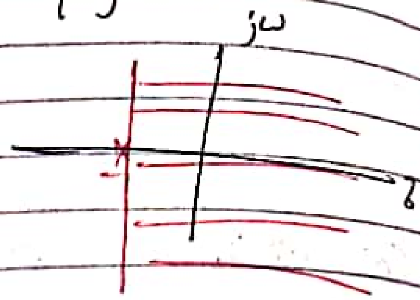



Example $h(t) = e^{-|t|}$ → two sided
→ system is non causal

$$H(s) = \frac{-2}{s^2-1}$$



**Example** $H(s) = \dfrac{e^{s}}{s+1}$ . $Re\{s\} > -1$




Is the system causal?

$$e^{-t}u(t) \longleftrightarrow \frac{1}{s+1}$$

Time shift

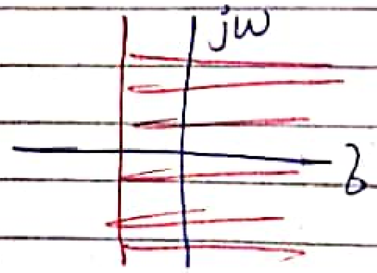
$$h(t) = e^{-(t+1)}u(t+1) \longleftrightarrow \frac{e^{s}}{s+1}$$

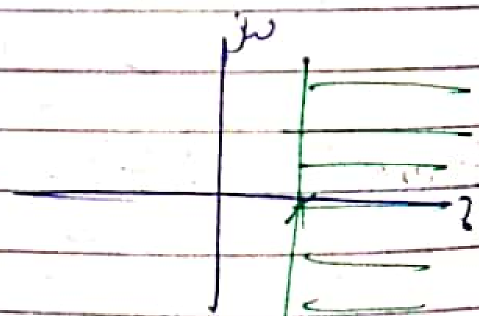
$$(h(t) = 0 \text{ for } t < 0) \; X$$

So the system is non causal. even though ROC is [illegible] in many cases [illegible] non causal

## Stability

An LTI system is stable if and only if the ROC of its system function includes the entire $j\omega$ axis ie $Re\{s\} = 0$

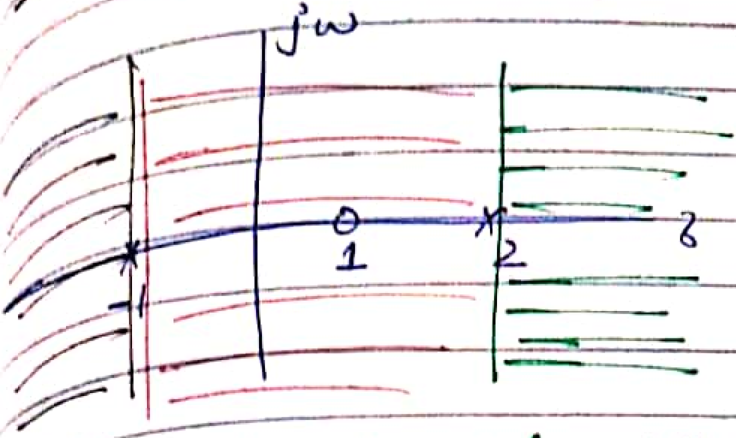



We know that $X(s)\big|_{s=j\omega} = F(x(t))$

$\Rightarrow$ $\boxed{\text{If the FT exists, the system is stable.}}$

Example $H(s) = \frac{s-1}{(s+1)(s-2)}$ ROC = ?



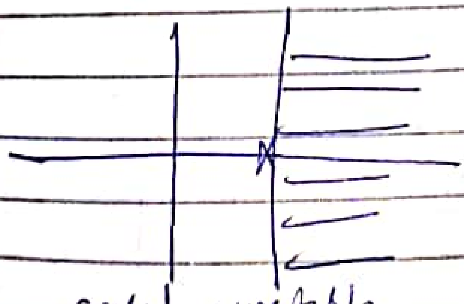

If system is causal; $Re\{s\} > 2$ $h(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t)$

If system is stable; $-1 < Re\{s\} < 2$ $h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$

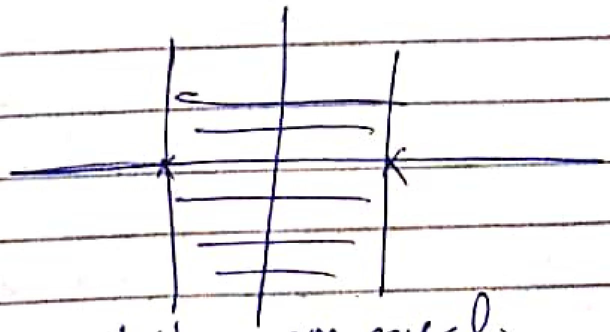
ROC: $Re\{s\} < -1$

unstable, Non causal

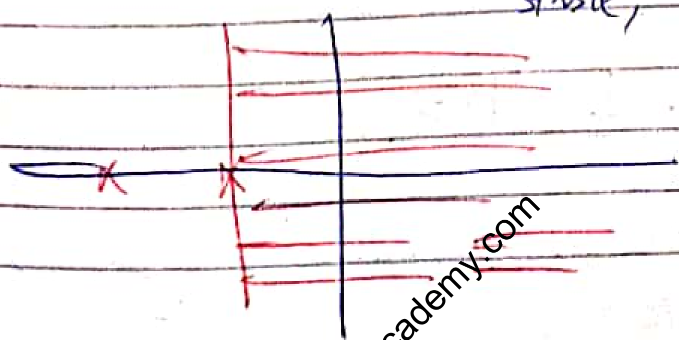
→ A Causal system with rational system function is stable if and only if all the poles of $H(s)$ lie in the left half of the s plane ie all poles have negative real parts.



causal, unstable



stable, non causal.



Stable + causal

# LTI Systems Characterized by LCCDE.

Say $\frac{dy(t)}{dt} + 3y(t) = x(t)$

$H(s) = ?$ $h(t) = ?$

If $x(t) = e^{st}$ $\rightarrow y(t) = H(s)\, e^{st}$

Put values of $x(t)$ and $y(t)$ in LCCDE and solve for $H(s)$.

Or <u>method 2</u> By taking Laplace transform;

$$L\left\{\frac{dy(t)}{dt} + 3y(t)\right\} = L\{x(t)\}$$

$$sY(s) + 3Y(s) = X(s).$$

$$Y(s)(s+3) = X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{(s+3)}$$

if non causal

if causal

$$\Rightarrow \boxed{H(s) = \frac{1}{s+3}}$$

ROC = ?

To find ROC, extra information is required. i.e about stability, causality etc.

## $N^{th}$ order LCCDE

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Apply LT;

$$\sum_{k=0}^{N} a_k \, L\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \, L\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\Rightarrow \sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\Rightarrow Y(s) \sum_{k=0}^{N} a_k s^k = X(s) \sum_{k=0}^{M} b_k s^k$$

$$\Rightarrow \boxed{\frac{Y(s)}{X(s)} = H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}}$$

ROC : extra information

---

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \rightarrow \text{Bilateral laplace transform}$$

Similarly we have;

$$\boxed{\mathcal{X}(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt \rightarrow \text{Unilateral laplace transform}}$$

① If $x(t) = 0$ for $t < 0$;

$$X(s) = \mathcal{X}(s)$$

② Two signals that differ for $t<0$, but are identical for $t \geq 0$;

$X(s)$ for both is ~~same~~ different

$\mathcal{X}(s)$ for both is same.

ROC for $X(s)$ ↳ Always Right Half plane

## Properties of ULLT $X(s)$

The rest properties are same as that of bilateral LT.

The different discussed;

## Differentiation Property

| Bilateral $X(s)$ | Unilateral, $X(s)$ |
|---|---|
| $x(t) \xleftrightarrow{L} X(s)$ | $x(t) \xleftrightarrow{UL} X(s)$ |
| $\frac{d\,x(t)}{dt} \xleftrightarrow{L} s\,X(s)$ | $\frac{d\,x(t)}{dt} \xleftrightarrow{UL} s\,X(s) - x(0)$ |
| $\frac{d^2 x(t)}{dt^2} \xleftrightarrow{L} s^2 X(s)$ | $\frac{d^2 x(t)}{dt} \xleftrightarrow{UL} s^2 X(s) - s\,x(0) - x'(0)$ |

$x'(0)$ → first derivative.

Why Laplace transform Unilateral?

→ It is better to use $X(s)$ for causal systems.

LTI described by LCCDE. (more applicable) Till when i/p is not applied o/p is zero.

system initial conditions zero @ Initial rest condition

i.e. $x(t) = 0$ for $t < 0$ → bilateral

so $y(t) = 0$ for $t < 0$

Non zero initial conditions; → unilateral

i.e. if $x(t) = 0$ for $t < 0$,

so $y(t) \neq 0$ for $t < 0$.

Example $\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$

$H(s) = ?$ $Y(s) = ?$ $y(t) = ?$

For $x(t) = \alpha\, u(t)$. initial rest condition

LT; Bilateral:

$s^2 Y(s) + 3s\, Y(s) + 2Y(s) = X(s)$

$\Rightarrow H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}$

$X(s) = \frac{\alpha}{s}$

$Y(s) = H(s)\, X(s) = \left(\frac{1}{(s+1)(s+2)}\right)\left(\frac{\alpha}{s}\right)$

$\Rightarrow Y(s) = \frac{\alpha}{(s+1)(s+2)s}$

$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$

Partial fraction; $y(t) =$

$e^{-t} u(t)$ $\frac{1}{s+1}$

$y(t) = e^{-t} u(t) - e^{-2t} u(t)$

**Case 2** System initial conditions are;

$$y(0^-) = \beta \quad , \quad y'(0^-) = \gamma$$

Non zero initial conditions;

Unilateral laplace transform;

$$\left[s^2 Y(s) - s\,y(0^-) - y'(0^-)\right] + 3\left[sY(s) - y(0^-)\right] + 2Y(s) = \frac{\alpha}{s}$$

Rearrange for $Y(s)$

$$\boxed{Y(s) = \frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)} + \frac{\alpha}{s(s+1)(s+2)}}$$

Zero i/p response

Zero state response

If at initial int $\beta = 0,\ \gamma = 0$